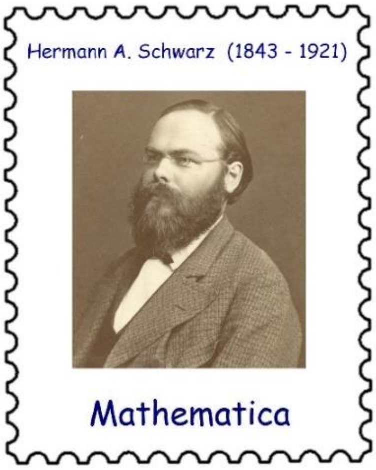
HERMANN AMANDUS SCHWARZ

(January 25, 1843 – November 30, 1921)

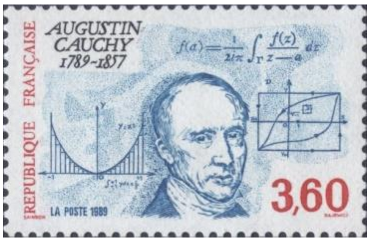
by HEINZ KLAUS STRICK, Germany



The name of the mathematician HERMANN AMANDUS SCHWARZ will probably be mentioned rather rarely in the context of lessons in the upper secondary school, although this would be quite conceivable.

The inequality $\left|\vec{u}\cdot\vec{v}\right| \le \left|\vec{u}\right|\cdot\left|\vec{v}\right|$ is called CAUCHY-SCHWARZ's inequality; sometimes the name BUNJAKOWSKI is added.

The inequality plays a role in geometry (see below), in analysis and also in statistics [$E(X\cdot Y)^2 \le E(X^2)\cdot E(Y^2)$].



The French mathematician AUGUSTIN-LOUIS CAUCHY proved the inequality for sums $\left(\sum_{i=1}^{n} u_i v_i\right)^2 \le \left(\sum_{i=1}^{n} u_i^2\right)\cdot\left(\sum_{i=1}^{n} v_i^2\right)$ in 1821.

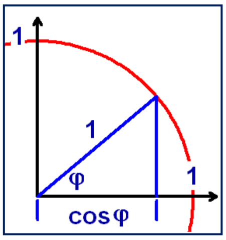
The Russian mathematician VIKTOR J BUNJAKOWSKI formulated it in 1859 for integrals of complex-valued functions; the general proof was given in 1888 by HERMANN AMANDUS SCHWARZ.

In vector calculus, the scalar or dot product $\vec{u} * \vec{v}$ of two vectors $\vec{u}, \vec{v}$ is defined as the sum of the products of the components of the two vectors: $\vec{u} * \vec{v} = u_1\cdot v_1 + u_2\cdot v_2 + u_3\cdot v_3$.

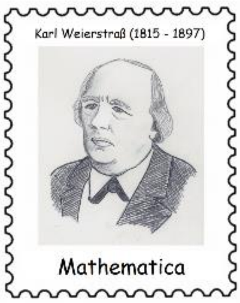
For the length of the vectors $\left|\vec{u}\right|^2 = u_1^2 + u_2^2 + u_3^2 = \vec{u} * \vec{u}$ and $\left|\vec{v}\right|^2 = v_1^2 + v_2^2 + v_3^2 = \vec{v} * \vec{v}$ is valid.

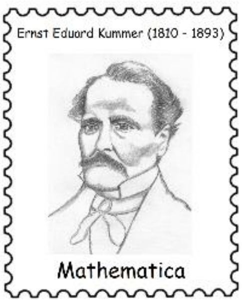
With the help of the theorem of PYTHAGORAS it can be proved that:
$\vec{u} \perp \vec{v} \Leftrightarrow \vec{u} * \vec{v} = u_1\cdot v_1 + u_2\cdot v_2 + u_3\cdot v_3 = 0$.

From this follows: $\vec{u}\cdot\vec{v} = \left|\vec{u}\right|\cdot\left|\vec{v}\right|.\cos(\varphi)$, and because of $0 \le \left|\cos(\varphi)\right| \le 1$ finally:
$\left|\vec{u} * \vec{v}\right| \le \left|\vec{u}\right|\cdot\left|\vec{v}\right| = \left(\vec{u} * \vec{u}\right)\cdot\left(\vec{v} * \vec{v}\right)$



After attending school at a grammar school in Dortmund, HERMANN AMANDUS SCHWARZ began to study chemistry at the Berlin *Gewerbeinstitut* (today: the Technical University). After attending the lectures of KARL WILHELM POHLKE, Professor of Descriptive Geometry at the Berlin *Bauakademie*, his interest in mathematics grew. In 1861 he also attended to lectures on integral calculus by KARL WEIERSTRASS at the University of Berlin.



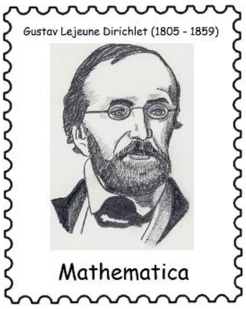


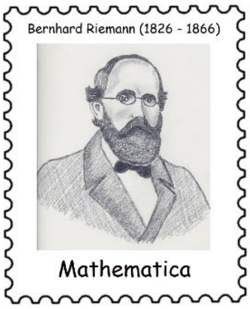
(drawings © ANDREAS STRICK)

Under the influence of Weierstrass and also Ernst Eduard Kummer, Schwarz switched to mathematics. As early as 1864 he completed his doctorate under Kummer with a thesis on *developable* surfaces (*De superficiebus in planum explicabilibus primorum septem ordinum*), i.e. surfaces that can be transformed into a plane without distorting their shape - examples of these are pieces of the lateral surface of a cylinder or a cone.

After completing his doctorate, Schwarz continued his studies in order to obtain a teaching qualification for secondary schools and at the same time he taught at various Berlin grammar schools.

In 1867 he received his *habilitation* and became a private lecturer at the University of Halle. In 1869 he was appointed to a professorship at the *Polytechnikum* in Zurich (today: the ETH). In 1875 he accepted an appointment to Göttingen to the post previously held by Carl-Friedrich Gauss, Gustav Lejeune Dirichlet and Bernhard Riemann, among others.





His main areas in which he researched and taught were complex analysis, differential geometry and calculus of variations (determination of maxima and minima of functions in multidimensional space).

Schwarz's contributions to various areas of mathematics were manifold. In particular they stimulated other mathematicians to further investigations:

- During his studies he had proved a theorem which Pohlke had suspected but could not prove exactly (the main theorem of *axonometry*). From this theorem followed, among other things:

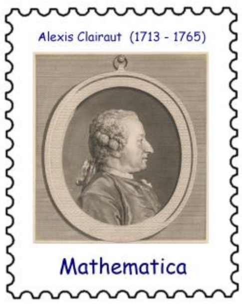
  Every quadrilateral lying in a plane can be understood as the image of a tetrahedron formed by a parallel projection.

- In analysis, he investigated, among other things, so-called conformal (angle-preserving) mappings. Schwarz's lemma is named after him, which states that for every complex differentiable function *f* with $f(0)=0$, which maps the open unit circle disk in the complex number plane onto itself, the following holds:

  $|f(z)| \le |z|$ as well as $|f'(0)| \le 1$.

- In 1740, Alexis-Claude Clairaut had assumed that the order in which the second partial derivative of a function of several variables is formed does not matter, i.e. that the following holds:

  $\frac{\partial}{\partial x}(\frac{\partial}{\partial y} f(x,y)) = \frac{\partial}{\partial y}(\frac{\partial}{\partial x} f(x,y))$.

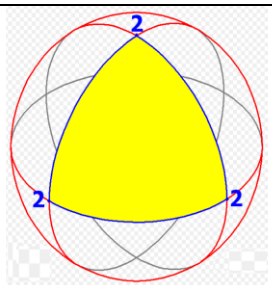
  

  Schwarz proved in 1873 that the continuity of the derivatives must be assumed, and he himself gave a counterexample:

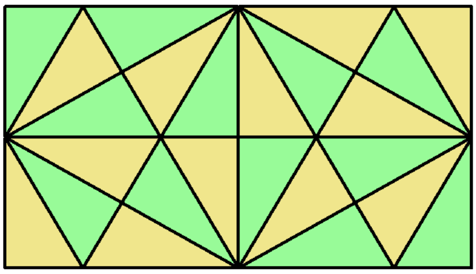
  $\frac{\partial^2}{\partial x \partial y}(\frac{x^3 y - x y^3}{x^2+y^2})(0,0) = 1$; $\frac{\partial^2}{\partial y \partial x}(\frac{x^3 y - x y^3}{x^2+y^2})(0,0) = -1$.

- In the same year, he published the classification of triangles with which the surface of the sphere, the Euclidean plane or the hyperbolic plane can be sub-divided (SCHWARZ triangles).

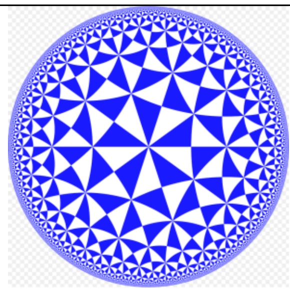
  For positive rational numbers p, q, r, triangles with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ exist on the sphere, if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$; in the (Euclidean) plane, if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$; and in the hyperbolic plane, if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$.



$p=q=r=2$



$p=6, q=3, r=2$

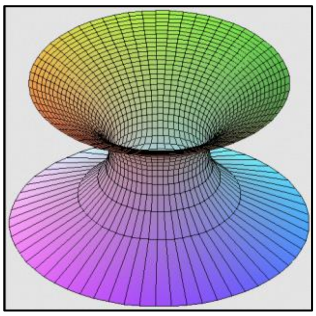


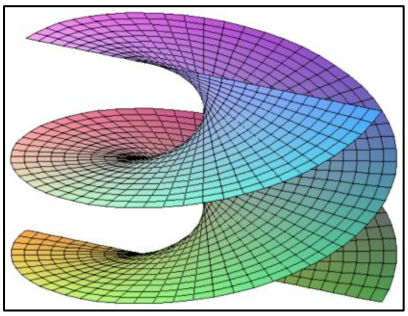
$p=5, q=4, r=2$

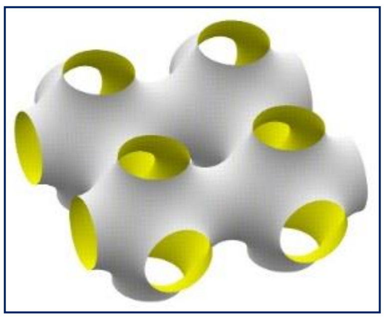
- Since the 1830s, the interest of mathematicians in the study of minimal surfaces had grown after the Belgian physicist JOSEPH ANTOINE PLATEAU had conducted soap-skin experiments and thus experimentally determined surfaces with minimal surface area.

  Prior to this, LEONHARD EULER had already made a first contribution to the subject in 1744 (a Catenoid, the figure below on the left) and in 1755 the 19-year-old JOSEPH-LOUIS LAGRANGE formulated the associated conditions (EULER-LAGRANGE equations).

  In 1776 JEAN-BAPTISTE MEUSNIER proved that the spiral surface (centre figure) satisfies these equations, and furthermore that the mean curvature must be zero for minimal surfaces. In the 1880s, SCHWARZ and his student EDVARD RUDOLF NEOVIUS investigated periodic minimal surfaces; the figure on the right shows a simple example.

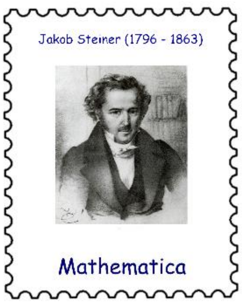






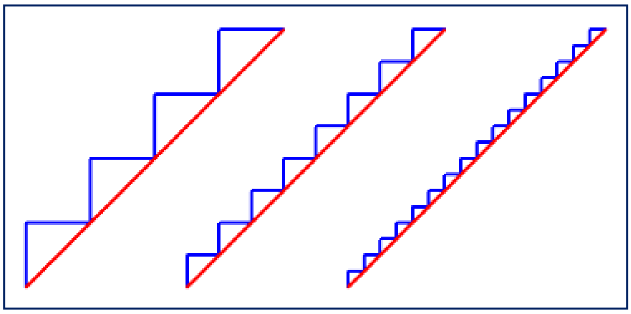
(graphics Wikipedia)

- In 1884 SCHWARZ completed JACOB STEINER's proof that among all bodies with a fixed surface, the sphere has the largest volume: In the set of continuously differentiable, simply closed, orientable surfaces of genus zero (that is, surfaces without holes), the sphere is the surface which, for a given surface, encloses the largest volume.



- On the occasion of the 70th birthday of his revered teacher Karl Weierstrass in 1885, Schwarz published the first complete proof (in the proverbial "Weierstrassian rigour") that minimal surfaces do indeed have minimal surface area.
- The so-called staircase paradox had been known for a long time: The approach to determine the length of a line by means of a line in the shape of a staircase does not result in the length of the line when the limit value of the length of the line is calculated (cf. the left-hand figure below).

  Schwarz developed a 3-dimensional analogue with the so-called Schwarz boot (cf. figures on the right): The lateral surface of a cylinder is approximated by rings of antiprisms of obtuse-angled isosceles triangles. With his example, he showed that the method of determining the area of curved surfaces with the help of polyhedron sequences contained in standard works on analysis until then was not sufficient.



After Weierstrass retired, Schwarz moved to his chair at the *Friedrich-Wilhelms-Universität* in Berlin (today: the Humboldt-Universität) in 1892, which he held until his own retirement in 1917. He was committed to supporting numerous students - as he had done previously in Göttingen.

His marriage to a daughter of Kummer produced six children. In his spare time, he loved to lead a brigade of the local volunteer fire brigade and to help the foreman of the neighbouring railway station with the dispatching of trains.

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https://www.spektrum.de/wissen/hermann-amandus-schwarz-der-herr-der-ungleichungen/2106303

Translated 2023 by John O’Connor, University of St Andrews

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