NIKOLAI LOBACHEVSKY (December 1, 1792 – February 24, 1856)

by HEINZ KLAUS STRICK, Germany

NIKOLAI IVANOVICH LOBACHEVSKY was born in Nizhny Novgorod. When his father died in 1799, his mother moved to Kazan on the central Volga. There he began studying medicine at the newly founded university at the age of 15, but switched to mathematics a year later.

His professor was MARTIN BARTELS, a friend of CARL FRIEDRICH GAUSS. He had previously worked as a teacher in Germany and was now appointed as a mathematics professor in Kazan. LOBACHEVSKY finished his mathematics studies at the age of 19 and at the age of 24 he was appointed professor at the University of Kazan. Later he became the dean and rector of the university.

At the age of 22 he was already concerned with the question of the meaning of the so-called axiom of parallels, that is, the fifth postulate of EUCLID's geometry. Many mathematicians had been concerned with this postulate since antiquity, but it was only LOBACHEVSKY, GAUSS and JANOS BOLYAI who achieved decisive new insights almost simultaneously. However, they were not recognised during their lifetime.

LOBACHEVSKY himself also received high honours, such as elevation to the hereditary nobility; however, this was not because of his discovery / invention of a new geometry. His *Geometric Investigations of the Theory of Parallels*, published in 1840, was viewed by most as the crazy ideas of an otherwise deserving scientist.

EUCLID had around 300 BC in the first volume of his *Elements* proposed a system of five postulates from which all geometrical propositions should be derived:

(1) Two points can always be connected by a line.

(2) A line can always be extended to a straight line.

(3) A circle is defined by specifying the center point and the radius.

(4) All right angles are equal to each other.

(5) If a straight line intersects two straight lines and forms inner angles with them on the same side, which together are smaller than two right angles, then the two straight lines intersect on the side on which the two angles lie, which are together smaller than two right angles.

It is noticeable that the fifth postulate differs significantly from the other four postulates in terms of the type of formulation. Many eminent mathematicians tried in vain to show that this postulate could be derived from the first four postulates. Over time, other equivalent systems of the postulates were discovered.

The 5th postulate can, for example, be replaced by:

- *For a straight line g and a point P that does not lie on the straight line, one can draw exactly one straight line that runs through P and is parallel to g.*

This formulation is sometimes referred to as the *axiom of parallels*. Other equivalent formulations are:

- *The sum of the interior angles of a triangle is 180°.*
- *The angles at which a line intersects parallel lines are the same.*

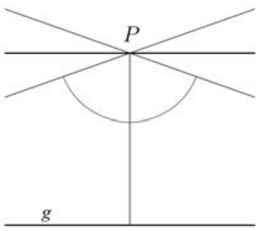
Gauss recognised around 1817 that the axiom of parallels could not be derived from the first four postulates and investigated the question of what kind of geometry would result if the fifth postulate was considered invalid.

Although he discussed his approaches with a friend, the Hungarian mathematician Farkas Bolyai, he was reluctant to publish his thoughts, as the philosopher Immanuel Kant had authoritatively stated a few years earlier in his *Critique of Pure Reason* that the geometry of Euclid is "essential": that is, irrevocably true.

However, Farkas Bolyai's son Janos Bolyai ignored his father's concerns and from 1823 developed a "new" geometry without the axiom of parallels.

In distant Kazan, Nikolai Ivanovich Lobachevsky – without knowledge of the investigations by Janos Bolyai - gave a lecture on an "imaginary" geometry in 1826:

- *For a given straight line g there should be at least two parallel straight lines that pass through a given point (and which do not meet g).*



In the following years he published several essays that were not noticed in western Europe. However, a contribution that appeared in French in 1837 drew the mathematical world's attention to the genius in the east. Gauss was so impressed by the work on *non-Euclidean geometry* (the name comes from Gauss) that he arranged for Lobachevsky to be appointed a corresponding member of the University of Göttingen.

Lobachevsky geometry is also referred to as *hyperbolic geometry*. The designation indicates that there is *more than one parallel* through every point. It contrasts with *elliptic geometry* where there is *no parallel* to a given line through a given point. In this notation, standard Euclidean geometry would be *parabolic geometry*.

The modified 5th postulate has surprising consequences:

- *In a triangle, the sum of the interior angles is always less than 180 °.*
- *If two triangles match at the angles, then they are congruent to each other.*

- *The set of all points which are at the same distance from a given straight line and lie in the same half-plane of this straight line does not itself form a straight line.*
- *You cannot always draw a circle through three points that are not on a straight line. etc.*

LOBACHEVSKY was highly respected as a university lecturer and rector.

His book on analysis contained a number of new insights and methods, including a procedure for the iterative determination of zeros of n^{th}-degree polynomials (the so-called DANDELIN-LOBACHEVSKY-GRÄFFE method – the method was also discovered independently about 1830 by the French mathematician GERMINAL PIERRE DANDELIN and by the German mathematician KARL HEINRICH GRÄFFE):

For example, if a third-degree polynomial has zeros r_1, r_2, r_3 with $|r_1| > |r_2| > |r_3|$ then:

$$f(x) = x^3 + a_2x^2 + a_1x + a_0 = (x - r_1)(x - r_2)(x - r_3)$$

with $a_2 = r_1 + r_2 + r_3$, $a_1 = r_1r_2 + r_1r_3 + r_2r_3$, $a_0 = -r_1r_2r_3$.

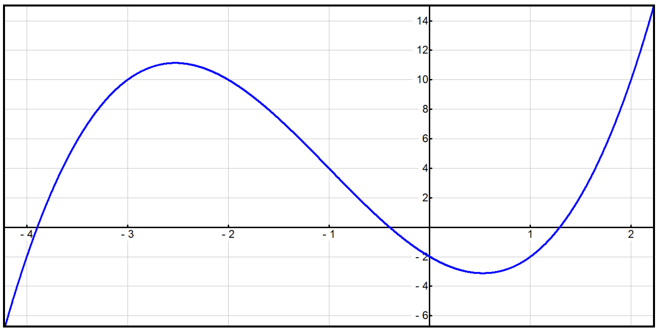
From this we get another 3^{rd} degree function, whose zeros are the squares of the original zeros:

$$f_1(x) = -f(\sqrt{x}) \cdot f(-\sqrt{x}) = (x - r_1^2)(x - r_2^2)(x - r_3^2) = x^3 + b_2x^2 + b_1x + b_0 \text{ with}$$

$$b_2 = -(r_1^2 + r_2^2 + r_3^2) = -a_2^2 + 2a_1,\ b_1 = r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2 = a_1^2 - 2a_2a_0,\ b_0 = -r_1^2r_2^2r_3^2 = -a_0^2$$

If the procedure is continued, the zeros will lie further and further apart as they are squared, and the coefficients of the polynomials will also increase in magnitude.

Example: $f(x) = x^3 + 3x^2 - 4x - 2$



From $f(x) = x^3 + 3x^2 - 4x - 2$ we get $f_1(x) = x^3 - 17x^2 + 28x - 4$, $f_2(x) = x^3 - 233x^2 + 648x - 16$ and $f_3(x) = x^3 - 52993x^2 + 412448x - 256$.

The quotients of successive coefficients provide increasingly better approximations of the zeros (their signs are obtained by trial and error):

- Approximations of the zeros from the polynomial $f_1(x)$:

$$-\frac{b_2}{b_3} = -\frac{-17}{1} = \frac{r_1^2 + r_2^2 + r_3^2}{1} \approx r_1^2,\quad -\frac{b_1}{b_2} = -\frac{28}{-17} = \frac{r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2}{r_1^2 + r_2^2 + r_3^2} = \frac{r_2^2 + r_3^2 + \frac{r_2^2r_3^2}{r_1^2}}{1 + \frac{r_2^2}{r_1^2} + \frac{r_3^2}{r_1^2}} \approx r_2^2,$$

$$-\frac{b_0}{b_1} = -\frac{-4}{28} = \frac{r_1^2r_2^2r_3^2}{r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2} = \frac{r_3^2}{1 + \frac{r_3^2}{r_2^2} + \frac{r_3^2}{r_1^2}} \approx r_3^2,\ \text{also } r_1 \approx -4.12,\ r_2 \approx 1.28,\ r_3 \approx -0.38.$$

- Approximations of the zeros from the polynomial $f_2(x)$:

$$r_1 \approx -\sqrt[4]{\frac{233}{1}} \approx -3.91,\ r_2 \approx \sqrt[4]{\frac{648}{233}} \approx 1.29,\ r_3 \approx -\sqrt[4]{\frac{16}{648}} \approx -0.40.$$

From the polynomial $f_3(x)$ the zeros are obtained with 3-digit accuracy

$$r_1 \approx -\sqrt[8]{52993} \approx -3.895;\ r_2 \approx \sqrt[8]{\frac{412448}{52993}} \approx 1.292;\ r_3 \approx -\sqrt[8]{\frac{256}{412448}} \approx -0.397.$$

First published 2006 by Spektrum der Wissenschaft Verlagsgesellschaft Heidelberg

https://www.spektrum.de/wissen/nikolai-lobatschewski-1792-1856/862792

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