Hardy As I Knew Him

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Let me begin by reminding you of some of the general facts about Hardy's career. Godfrey Harold Hardy was born in 1877, was fourth Wrangler in 1898, coming after R.W.H.T. Hudson, J.H. Jeans, and J.F. Cameron, and became a fellow of Trinity College, Cambridge, in 1900. In 1919, he was elected to the Savilian chair of Geometry in Oxford and so became a fellow of New College; he returned to Cambridge in 1931 as Sadlerian Professor of Pure Mathematics, again with a fellowship at Trinity. He retired in 1942 and died in 1947.

His mathematical work was mainly in analysis and number theory. His famous collaboration with J.E. Littlewood began in 1912, and that with Srinivasa Ramanujan lasted from 1913 to 1919.

My own contacts with Hardy began in 1933. I had arrived in Cambridge in 1931 as an affiliated student with an Edinburgh degree, and could therefore sit the Tripos and take the B.A. degree after two years; at that time, Part II of the Tripos consisted of Schedules A and B, corresponding approximately to the present Parts II and III. In the Lent Term of 1933, I attended a course of lectures by Hardy on Fourier series and orthogonal expansions, and decided that I wanted to start research under his supervision. I went to see him and asked if he would take me on; he asked what field I wanted to work in. I told him that I was interested in integral equations, about which I already knew a little; his reaction was, "Good, it's time I learned something about them". In fact, he had published several papers on special integral equations, but he had never been much concerned with the general theory.

He suggested several problems to me, and I selected differential equations of fractional order, an investigation that developed into some work on the non-essentially singular integral equations of Volterra type. His method of supervision was written rather than oral; I would take a piece of work for him to look at, and a day or two later he would send me written comments on it, making suggestions for further developments. It may be recalled that his collaboration with Littlewood worked in much the same way. He would always ask whether results were 'best possible', a favourite phrase of his; such queries usually led to the construction of counter-examples to show the conditions could not be further improved. Once one had arrived at 'best possible' results, his next question was, "What happens when the conditions are no longer satisfied; can one get any results then, and how do they differ from the initial ones?". This kind of prodding provided a superb training in analysis.

His attitude to the Ph.D. degree was somewhat dismissive; what he said to me, rather grumpily, at the beginning of my research career, was, "I suppose you may as well register as a candidate for the degree; it seems to be the fashion nowadays". The degree was comparatively new in Cambridge, having been set up only about 1922. He and J.M. Whittaker were my examiners for the degree in 1936; the oral examination was held beside the Trinity bowling green and was little more than a formality. He asked me, "Suppose that you were asked at a moment's notice to give a course on Fourier series; where would you begin?". My rejoinder that I would commence with the L^2 theory seemed to content him; after some trivial questions about my thesis, he handed over to Whittaker, who merely asked me for advice about the literature on integral equations.

Between 1933 and 1936, I attended Hardy's courses regularly. His subjects included: trigonometric series and integrals; number theory; orthogonal systems of functions; the calculus of variations; and divergent series. He was an excellent lecturer, and one got good notes of what he expounded; but they had to be gone through afterwards, for he would often omit part of a proof, with the remark, "any competent analyst can put in the epsilons". At the beginning of his course on the calculus of variations, he surveyed the literature on the subject; his comment on Forsyth's book was, "In this enormous volume, the author never succeeds in proving that the shortest distance between two points is a straight line".

In the course on Fourier transforms, he said that every analyst had his own pet proof of Plancherel's theorem, and then proceeded to give four different ones. He was quite right; I had (and still have) my own pet proof, which is different from all four.

In his course on orthogonal systems of functions, Hardy included some of the general theory of integral equations, developing a Lebesgue-integral version of the Erhard-Schmidt theory for symmetric kernels; he found it trickier than he had expected, and suggested one or two erroneous results on mean-square convergence, to which I was able to provide counter-examples.

He usually lectured easily and confidently. The only time I have seen him nervous in a lecture was an occasion when Landau walked in and sat down at the back of the room; that day Hardy was very hesitant and fumbled some of his proofs – perhaps he was remembering the Göttingen tradition of vigorous heckling. However, on another occasion, he summed up Landau's position in mathematics very aptly; he said, "Landau will never prove the Riemann hypothesis, but, if anyone else does, Landau will have a better proof inside a week.".

Everyone concerned with analysis or number theory regularly attended the 'Hardy-Littlewood Conversation Class'. This was a weekly seminar that Littlewood had founded; when Hardy came back to Cambridge, it continued to meet in Littlewood's rooms in Trinity until 1934, but Littlewood himself gradually faded out; when I was attending it in 1933-34, he was never seen. A quite elaborate tea was served first, and then someone would give a talk; each meeting was announced by postcard giving details of the speaker and his topic, and sent by post. From 1934 onwards the meetings moved to the Arts School in Bene't Street, but the tradition of the tea beforehand continued. The other main pure mathematics seminar in Cambridge at that time was 'Baker's tea party', held for the geometers in H.F. Baker's house on Storey's Way

Speakers at the Conversation Class might be senior members of the university, research students or visitors. Many refugees from Nazism turned up in Cambridge during that period; among them were Richard Courant, Fritz John, Richard Rado, Hans Heilbronn, Werner Rogosinski, Olga Taussky, Bernhard Neumann, and (later) John von Neumann, Hans Hamburger, and Nachman Aronszajn. All of them gave talks, or even courses of lectures: Courant gave a brilliant two-term course on partial differential equations and ran a small seminar of his own for a couple of terms; Rogosinski lectured on Fourier series; and von Neumann on operator theory. Other visitors included D.V. Widder and later, in 1938-39, André Weil, Beniamo Segre, Ralph Boas, D.C. Spencer, and D.H. Lehmer. Hardy did a great deal to help the refugee mathematicians, finding posts for many of them. The writing of the Hardy-Rogosinski tract on Fourier series was partly intended to provide some financial help for Rogosinski; Hardy did the final writing up, and was *rather childishly* proud of the fact that it worked out at exactly 100 pages and contained exactly 100 theorems.

On one occasion Hardy gave a talk to the Conversation Class under the title 'How not to write mathematics papers'. He began with an exposition of the technique of printing mathematics from movable type, describing the complications that can be caused to the printer by things like subscripts and subsubscripts. He then went into the questions of style, insisting that a mathematical equation is a sentence of the English language and should be punctuated accordingly. He even managed to find about a dozen faults of style or syntax in the postcard announcing the meeting (prepared by Davenport). He remarked on the ambiguity of words like 'as' or 'any', and warned against beginning a sentence with 'Now'; he called this bullying the reader. He wound up with examples of three types of bad English commonly found in mathematical papers. One was Polish English, usually from Polish or Japanese authors, and characterised by misusing or not using definite and indefinite articles. Another was German English, of which a characteristic feature was the attempt to squeeze all qualifying conditions into the same sentence; some English or American authors slipped into this, an example being E.W. Hobson. Finally there was schoolboy English, of which he gave as an example:

1 pig = £5,
$\therefore$ 4 pigs = £20 = Ans.

I cannot resist telling of another incident at the Conversation Class, though it had nothing to do with Hardy. Not all the talks there were good ones; on this occasion we had an extraordinarily dull one on singularities of analytic functions, which provoked Ralph Boas into writing a clerihew:

Is there anything lowlier
Than the singularities of Polya?
By the time they are classified
The audience will be ossified!

When Hardy returned to Cambridge in 1931, his rooms were in Whewell's Court, at Trinity. When I visited him there, I could not help noticing the three photographs on his mantelpiece, of people whom he regarded as being supreme in their respective spheres; they were of Einstein, Lenin, and the cricketer, Jack Hobbs. His highest praise of anyone was to say that they were 'in the Hobbs class'. Later, however, when Don Bradman came on the scene, he had to revise his classification, so Hobbs was demoted, and the Bradman class became tops. In about 1933, he moved to rooms in the Great Court, behind the clock tower, and this set he occupied for the rest of his life. When he became ill towards the end of his life, he was disturbed by the striking of the clock; I was told that the College arranged that the clock should no longer strike during the night.

In 1936, I went to the Institute for Advanced Studies, at Princeton, to work with von Neumann. Hardy was invited that year to the tercentenary celebrations at Harvard, where he lectured on Ramanujan. He came on to Princeton afterwards, staying for some weeks. When he was in the States his devotion to baseball almost equalled his love of cricket. I remember an occasion in the Graduate College common room when he and L.P. Eisenhart had a long session discussing fine points of the baseball rules. When war broke out in 1939, the British newspapers were abruptly reduced in size, and I recall Hardy being distinctly annoyed when *The Times* ceased reporting the games of the World Series, which was then in progress.

That autumn term of 1936 gave us plenty of other subjects to talk about in the common room. We had full reports of the events leading up to the abdication of Edward VIII – on the actual day of the abdication *The New York World-Telegram* devoted its first fourteen pages to the subject. Roosevelt was running for his second term as President, and, when the election results were coming through on the radio, the group gathered to listen comprised about a dozen Europeans and perhaps one American.

Politically Hardy was vaguely left-wing, though he was not a member of any party; he is best described as a follower of Bertrand Russell, who had influenced him strongly in his younger days. He was President of the Association of Scientific Workers for a couple of years. He remained a pacifist in principle throughout; he found it somewhat embarrassing when I asked him for a testimonial in my search for a war-time job, but he produced one saying nothing about my qualifications for war-related work.

Hardy's sports included cricket, squash, and real tennis. Besides these, Hardy liked walking, and sometimes undertook a walking tour for a holiday. He carried little, posting a parcel each day or two to his next destination, which indicates the reliability of the postal services in those days. On one occasion in the States, he was staying in New York. At that time there was a well known railroad junction called Manhattan Transfer, in the middle of the New Jersey marshes (the train conductor's cry as we approached it was, "Manhattan Tranfer, change for Joisey City, Hudson Toiminal, and dern-tern Noo Yawk"). Hardy fancied a walk; he took the ferry across the Hudson, and set out to walk to Manhattan Transfer. According to the story I was told, he was picked up by the police as a suspicious character – they could not believe that anyone would wish to walk to Manhattan Terminal for pleasure.

After I returned from the States, I used to look in on him from time to time; if he was not in his rooms, he was likely to be sunning himself in the Trinity bowling green. On one occasion I found him sitting there correcting the proofs of his book *Divergent Series*; he complained that once one had read through a passage three times, it became gospel, and one became quite unable to spot any error in it. I continued to visit him during the war when I could manage to get away for the weekend in Cambridge; when he was writing the *Apology*, he described it to me as a gloomy book.

Another incident comes to mind. It is well known that Hardy regarded God as his personal enemy; he would not enter a religious building, even refusing to go into his College chapel to take part in the election of a new head of College – special provision had to be made to enable him to cast his vote. On one occasion, he was due to give a talk to the Adams Society, as the mathematical society at my College, St. John's, is known, and I invited him to dine in College beforehand. We have a very long after-dinner grace. Throughout it, Hardy stood with hands in in his trouser pockets, jingling his keys loudly – probably quite unconsciously.

During the war, I had a letter from Ralph Boas, mentioning that Hardy had once said to him that modern mathematicians were unable to evaluate definite integrals, a skill in which Hardy excelled. Ralph went on to say that there must be some truth in this, for he had been defeated by the definite integral

$$\int_0^\infty e^{-\sqrt{1+x^2}} dx.$$

When the letter arrived, I was due to attend a very dull meeting about shell production, which was not really my business anyway. So, I spent most of the meeting playing around with the integral, and got far enough to realize that it would be expressible it terms of Bessel functions. I finished it off when I got back to my lodgings.

For many years, Hardy was one of the editors of the Cambridge Mathematical Tracts. When I came back to Cambridge at the end of the War, he invited me to write a Tract on integral equations to replace Bôcher's Tract of 1909. I took him up on this, but my Tract did not appear until 1958.

Hardy was very sceptical about the value of abstract and axiomatic methods in mathematics. He once drew a distinction between mathematics in Cambridge and Princeton: Princeton mathematics, he said, was like constructing a lot of interconnecting subways just below the surface; Cambridge mathematics, in contrast, was like digging a deep hole straight down. He did not believe that the abstract methods could get a result in classical analysis unobtainable by more traditional methods. However, in spite of this attitude, when I had an interview with him at the home of E.T. Whittaker in Edinburgh, on the occasion he was there to receive an honorary degree and before I started research, he did suggest that I should read Banach's book on linear operators and Stone's book on Hilbert space, both of which had come out the previous year.

Just after the end of the War, I was able to shake Hardy's scepticism a little. Douglas Northcott had been a pupil of Hardy immediately before the War, but had joined the Army after one year, ending up as a Japanese prisoner-of-war. When he came back after the war, he decided to switch to me as a supervisor, having become interested in functional analysis. He proved some Tauberian theorems for vector valued functions. One of these results implied that, if (a_n) is a real valued sequence such that

$$a_n = O(n^{\gamma/2-1})$$

where $\gamma > 0$, and we write

$$\phi(u) = e^{-u/2}(a_0 + a_1 e^{-u} + a_2 e^{-2u} + \ldots),$$

and

$$J = \frac{1}{\Gamma(\gamma)} \int_0^\infty u^{\gamma-1}(\phi(u))^2 du,$$

then

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \frac{a_\mu a_\nu}{(\mu+\nu+1)^\gamma} \to J \quad \text{as } m, n \to \infty.$$

This implication holds irrespective of whether J is finite or not. The special case when $\gamma = 1$ tells us that if $a_n = O(n^{1/2})$, then the Hilbert double sum

$$\sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{a_\mu a_\nu}{(\mu+\nu+1)},$$

which is well known to be convergent when $\sum a_n^2$ is finite, either converges or diverges to $+\infty$.

I told Hardy about this result, and, after a while, he reported that he had managed to prove the special case with some difficulty, but had found no general means of attack.

References