# A transcription of Tait's notes on Terrot's lecture 

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> On the
> Imaginary roots of Negative Quantities. By the Right Reverend Bishop Terrot. $\underline{1847}$

1. $\sqrt{-1}$ is called impossible or imaginary $\because$ no ordinary algebraic quantity which must be either + or - can give when squared a negative result. Considering however the common application of Algebra to Geometry we easily see, that the assumption that every line must be either + or - is inconsistent with the possibility of drawing a line in any direction. $+1 \times a$ means a line whose length is $a$ drawn in one direction, $-1 \times a$ means the same length of line but drawn in a different direction, and to say that a line of the length of $a$ cannot be drawn in any other direction than one of these is absurd. $\sqrt{-1}$ $\therefore$ is not impossible any more than - or +1 and shows only the direction of the line to which it is affixed.
2. If from $C$ [See Fig 1, Lewis] we draw any number of lines such that they shall be in continued proportion and make at the same time $\angle A C A_{1}=A_{1} C A_{2}=A_{2} C A_{3}$ \&c then calling $C A=1, C A_{1}=a, C A_{2}=a^{2}$ or the lines are in this series $a^{0}, a^{1}, a^{2}, a^{3} \& \mathrm{c}$ while the angles which they make with the line $C A$ are $0, \vartheta, 2 \vartheta, 3 \vartheta \& c$ being the angle $A C A_{1} \times$ exponent of that radius vector ( $C A_{a}$ for example) from which to $C A$ they are measured. Thus the line whose angle of inclination is on $n \vartheta$ has its length $=a^{n}$ \& vice versâ.
3. If we now assume the several lines $C A, C A_{1}, C A_{2}, \& c$ [See Fig 2, Lewis] all equal or radii of a circle the case will not be altered. Let $n$ be a divisor of $2 r \pi$ or let $\vartheta=\frac{2 r \pi}{n}$. Thus the Radius $a^{n}=a^{\frac{2 r \pi}{\vartheta}}$ is the same in length \& position as CA $\therefore a^{1}=1^{\frac{1}{n}}=1^{\frac{\vartheta}{2 r \pi}}$. We know from ordinary Algebraical principles that the several $n$th roots of unity may be expressed by the series $a, a^{2}, a^{3}, \& c$. It therefore follows that we may take the successive Radii of a circle at equal angles for the several roots of unity $\&$ conversely. If $R$ be the numerical length of radius that radius inclined to the first at $\angle \vartheta$ is $=R \times 1 \frac{\vartheta}{2 r \pi}$. We $\therefore$ call $1 \frac{\vartheta}{2 r \pi}$ the coefficient of direction because it refers only to the direction, never to the length of a line. Thus, $a \times \frac{1+\sqrt{-3}}{2}$ is a line $=a$ simply.
4. Let us next suppose $n=2, A B$ will be a diameter \& if $C A=1, C B=-1$. But $a^{2}=1 \therefore a= \pm 1$. But the radii being $a, a^{2}, a$ must evidently be $=-1 \& a^{2}=+1$.

Next let $n=4, C A, C D, C B, C E$ are the 4 roots of the equation $a^{4}-1=0$. But the roots are $\pm 1 \& \pm \sqrt{-1}$. Here $C A \& C B$ are symbolized by $+1 \&-1$ respectively $\therefore$ $C D \& C E$ must be symbolized by $+\sqrt{-1} \&-\sqrt{-1}$ respectively, it being however quite optional which direction from $C$ we account positive or negative either in the horizontal or perpendicular lines.
5. It appears from the foregoing Props. that if a line is symbolised by $=a \cdot 1 \frac{\vartheta}{2 r \pi}$ we know both its length \& direction. $a \cdot 1 \frac{\vartheta}{2 r \pi} \therefore$ represents the actual transference of the point in space by moving from $A$ to $C$. [See Fig 3, Lewis] But it is also clear that its actual transference in space though not its distance travelled would be the same did it move from $A$ to $B \&$ then from $B$ to $C$. Thus $\therefore(A C \times$ its coefficient of direction $)=$ $(A B \times$ its coefficient of direction $)+(B C \times$ its coefficient of direction $)$. Therefore also the sum of any two lines making an angle with each other is = the diagonal of their parallelogram completed. Even in this startling form it is only the general assertion of a proposition particular cases of which we admit when we say $A B_{1}+B_{1} C=A C$ or that $A C+C B_{1}=A B_{1}$.

1. As examples to elucidate this let $A B C$ (Fig 4) [See Fig 3, Lewis] be an isosceles right angled triangle described on the radius $A D$. If we call $A B$ the radius or Hypotenuse $a$ each of the sides will be in length $\frac{a}{\sqrt{2}} \& A B$ is symbolized by $a \times 1^{\frac{45}{360}}=a \times 1^{\frac{1}{8}}=a \times \frac{1+\sqrt{-1}}{\sqrt{2}}$. But $A C=\frac{a}{\sqrt{2}}$. $C B$ being perpendicular to original position is $=\frac{a}{\sqrt{2}} \times \sqrt{-1}$ (Prop. 4) $\therefore$ $A C+C B=a \times\left[\frac{1}{\sqrt{2}}+\frac{\sqrt{-1}}{\sqrt{2}}\right]=a \times \frac{1+\sqrt{-1}}{\sqrt{2}}=A B$.
2. Let $B A C=60^{\circ}, B C A=90^{\circ}$, then $A B$ in length \& direction is $a \cdot 1^{\frac{60}{360}}=a \cdot 1^{\frac{1}{6}}=$ $a \cdot \frac{1+\sqrt{-3}}{2}, A C=\frac{a}{2}, C B$ in length $=a \cdot \frac{\sqrt{3}}{2} \therefore$ in length \& direction jointly $=a \cdot \frac{\sqrt{3} \sqrt{-1}}{2}=a \cdot \frac{\sqrt{-3}}{2}$ $\therefore A C+C B=\frac{a}{2}+a \cdot \frac{\sqrt{-3}}{2}=a \cdot \frac{1+\sqrt{-3}}{2}=A B$.
3. Let the triangle (Fig 5) [See Fig 3, Lewis] be Equilateral \& let $A B$ be the original position. Let $A B=a, A C=a \cdot 1^{\frac{1}{6}}, C B=a \cdot 1^{\frac{-1}{6}} \therefore A C+C B=a \cdot\left[1^{\frac{1}{6}}+1^{\frac{-1}{6}}\right]=$ $a \cdot\left[1^{\frac{1}{6}}+\frac{1}{1^{\frac{1}{6}}}\right]=a \cdot\left[\frac{1^{\frac{1}{3}}+1}{1^{\frac{1}{6}}}\right]=a \cdot\left[\frac{-1+\sqrt{-3}}{2}+1\right] \times \frac{2}{1+\sqrt{-3}}=a \cdot\left[\frac{1+\sqrt{-3}}{2}+\frac{2}{1+\sqrt{-3}}\right]=a=A B$
4. In the foregoing Props. \& Examples it has been taken for granted that we know not only the several $n$th roots of unity but also their proper order; that is the order in which as coefficients they express the radii drawn to the extremities of the arcs $\vartheta, 2 \vartheta, 3 \vartheta$, \&c. with the original radius. But when we determine the roots of $x^{n}-1=0$ we obtain them in no fixed order. To discover this order we must observe that two roots are always of the form $a \pm \sqrt{-b}$ comparing which with (Fig 6) [See Fig 4, Lewis] $a$ is evidently the part symbolical of the cosine $+\sqrt{-b}$ that of the sine because it is affected by $\sqrt{-1}$ and is $\therefore$ perpendicular to original radius. Thus $\therefore$ in $a \pm \sqrt{-b},+$ refers to radii in the upper semicircle \& - to those in the under; and the two radii whose symbols differ only in the sign of $\sqrt{-b}$ are at equal angles to the original radius on opposite sides of it. $\therefore$ the root
in which $a$ is greatest is nearest to the original radius. Thus the roots of $n^{6}-1$ arranged properly are

$$
1, \frac{1+\sqrt{-3}}{2}, \frac{-1+\sqrt{-3}}{2},-1, \frac{-1-\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}
$$

symbolizing the radii drawn respectively to the ends of the arcs

$$
0^{\circ} \text { or } 360^{\circ}, 60^{\circ}, 120^{\circ}, 180^{\circ}, 240^{\circ}, 300^{\circ}
$$

For if +1 be first -1 having no sinal part must be in the middle. Next $\frac{1+\sqrt{-3}}{2} \& \frac{-1+\sqrt{-3}}{2}$ must be in the upper half of the circle and $\frac{1+\sqrt{-3}}{2}$ must come first because its cosine is in $C A$. And so with the rest.
7. It appears from Props. 4, 5 that the radius drawn to the end of an arc $\vartheta$ is $=\frac{\vartheta}{2 r \pi}$ and this again by $a \pm \sqrt{-b}$ where $a$ is what is trigonometrically called the cosine $\& \sqrt{b}$ the sine of $\vartheta$. Now (Fig 6) [See Fig 4, Lewis] let $\angle A C A_{1}=\vartheta, \angle A C A_{2}=2 \vartheta$, \&c $\angle A C A_{p}=p \vartheta$, then

$$
\begin{aligned}
& C A_{1}=C D+\sqrt{-1} \cdot D A_{1}=\cos \vartheta+\sqrt{-1} \cdot \sin \vartheta \\
& C A_{p}=\cos p \vartheta+\sqrt{-1} \cdot \sin p \vartheta
\end{aligned}
$$

But by prop. 2,

$$
\begin{aligned}
& C A_{p}={\overline{C A_{1}}}^{p}=(\cos \vartheta+\sqrt{-1} \cdot \sin \vartheta)^{p} \\
& \therefore(\cos \vartheta+\sqrt{-1} \cdot \sin \vartheta)^{p}=\cos p \vartheta+\sqrt{-1} \sin p \vartheta, \text { which is }
\end{aligned}
$$

Demoivre's Theorem.
cor. If $p \vartheta=2 \pi, \cos p \vartheta+\sqrt{-1} \cdot \sin p \vartheta=1$.
Hence $(\cos \vartheta+\sqrt{-1} \cdot \sin \vartheta),(\cos 2 \vartheta+\sqrt{-1} \cdot \sin 2 \vartheta) \& c$. represent the several $p$ th roots of unity. If we arrange the angles, instead of $\vartheta, 2 \vartheta, 3 \vartheta \& \mathrm{c}$, in pairs thus $\vartheta \& \overline{p-1} \cdot \vartheta, 2 \vartheta$ $\& \overline{p-2} \cdot \vartheta \& c$. the several expressions for $x$-the several $p$ th roots of unity or the simple factors of $x^{p}-1=0$ taken in pairs corresponding with the above will be

$$
\begin{aligned}
& (x-\cos \vartheta-\sqrt{-1} \cdot \sin \vartheta) \&(x-\cos \overline{p-1} \vartheta-\sqrt{-1} \cdot \sin \overline{p-1} \vartheta) \\
& \text { which last is }=(x-\cos \overline{p \vartheta-\vartheta}-\sqrt{-1} \cdot \sin \overline{p \vartheta-\vartheta})= \\
& (x-\cos \overline{2 \pi-\vartheta}-\sqrt{-1} \cdot \sin \overline{2 \pi-\vartheta})=(x-\cos \vartheta+\sqrt{-1} \cdot \sin \vartheta)
\end{aligned}
$$

In the same way the next pair must be

$$
(x-\cos 2 \vartheta+\sqrt{-1} \cdot \sin 2 \vartheta) \&(x-\cos 2 \vartheta-\sqrt{-1} \cdot \sin 2 \vartheta)
$$

Multiplying these together for the quadratic factors of $x^{p}-1$, we obtain when $p$ is even $x^{p}-1=\left(x^{2}-1\right)\left(x^{2}-2 x \cos \vartheta+1\right) \cdot\left(x^{2}-2 x \cos 2 \vartheta+1\right)$ to $\frac{p}{2}$ terms

But when $p$ is odd

$$
x^{p}-1=(x-1)\left(x^{2}-2 x \cos \vartheta+1\right) \& c \text { to } \frac{p+1}{2} \text { terms }
$$

where $\vartheta$ it may be observed is $=\frac{2 \pi}{p}$
8.

$$
\begin{aligned}
& \sin \overline{A+B}=\sin A \cdot \cos B+\cos A \cdot \sin B \\
& \cos \overline{A+B}=\cos A \cdot \cos B-\sin A \cdot \sin B
\end{aligned}
$$

Let $\operatorname{arc} A B($ Fig 7$)[$ See Fig 5 , Lewis $]=\mathcal{A}, B D_{2} \& A D_{1}$ each $=\mathcal{B}$.
Then by Prop. 3, $C B=r \cdot 1 \frac{\mathcal{A}}{2 \pi}, C D_{1}=r \cdot 1^{\frac{\mathcal{B}}{2 \pi}}, C D_{2}=r \cdot 1^{\frac{\mathcal{A}+\mathcal{B}}{2 \pi}}$

$$
\therefore C D_{2}=r \cdot 1^{\frac{A}{2 \pi}} \cdot 1^{\frac{B}{2 \pi}}
$$

But by Prop. 7,

$$
\begin{aligned}
& 1 \frac{\mathcal{A}}{2 \pi}=\cos \mathcal{A}+\sqrt{-1} \cdot \sin \mathcal{A} \\
& 1^{\frac{\mathcal{B}}{2 \pi}}=\cos \mathcal{B}+\sqrt{-1} \cdot \sin \mathcal{B}
\end{aligned}
$$

$\therefore 1^{\frac{\mathcal{A}+\mathcal{B}}{2 \pi}}=\cos \mathcal{A} \times \cos \mathcal{B}-\sin \mathcal{A} \times \sin \mathcal{B}+\sqrt{-1}(\sin \mathcal{A} \cdot \cos \mathcal{B}+\cos \mathcal{A} \cdot \sin \mathcal{B})$,
but $1^{\frac{\mathcal{A}+\mathcal{B}}{2 \pi}}=\cos \overline{\mathcal{A}+\mathcal{B}}+\sqrt{-1} \sin \overline{\mathcal{A}+\mathcal{B}}$.
Equating then the sinal and cosinal parts of these, we have,

$$
\begin{aligned}
& \cos \mathcal{A} \cdot \cos \mathcal{B}-\sin \mathcal{A} \cdot \sin \mathcal{B}=\cos \overline{\mathcal{A}+\mathcal{B}} \\
& \sin \mathcal{A} \cdot \cos \mathcal{B}+\cos \mathcal{A} \cdot \sin \mathcal{B}=\sin \overline{\mathcal{A}+\mathcal{B}}
\end{aligned}
$$

## Definition

It should be observed that in the following propositions a line expressed by letter simply as $A B$ must be considered both as to length \& direction while when in brackets thus $(A B)$ its length alone is referred to. Thus $(A B) 1^{\frac{\theta}{2 \pi}}=A B$.
9. In any right angled triangle the sum of the squares of the sides is $=$ square of hypotenuse.

Let $C A($ Fig 6$)[$ See Fig 4, Lewis $]=r$, then $C A_{1}=r \cdot 1^{\frac{\vartheta}{2 \pi}}, \& C A_{n-1}=r \cdot 1^{\frac{-\vartheta}{2 \pi}}$

$$
\therefore C A_{1} \times C A_{n-1}=r^{2} \times 1^{\frac{\vartheta}{2 \pi}} \times \frac{1}{1 \frac{v}{2 \pi}}=r^{2},
$$

Also $C A_{1}=\left(C D_{1}\right)+\sqrt{-1}\left(D_{1} A_{1}\right)$
$C A_{n-1}=\left(C D_{1}\right)-\sqrt{-1}\left(D_{1} A_{1}\right)$ for $\left(D_{1} A_{1}\right)=\left(D_{1} A_{n-1}\right)$
$\therefore C A_{1} \times C A_{n-1}=\left(C D_{1}\right)^{2}+\left(D_{1} A_{1}\right)^{2}$ which is $\therefore=r^{2}=(C A)^{2}=\left(C A_{1}\right)^{2}$
its equivalent in area.

## 10. Cotes' Properties of the Circle.

Let the circumference be divided into $n$ equal parts and join $O P_{1}, O P_{2}, O P_{3}$, \&c (Fig 8) [See Fig 6, Lewis] and also join $P_{1}, P_{2}, P_{3}$ with $C$ any point in the Diameter. Then

$$
\begin{aligned}
& C P_{1}=O P_{1}-O C, C P_{2}=O P_{2}-O C \& c \\
& \therefore C P_{1} \cdot C P_{2} \cdot C P_{3} \cdots C P_{n}=\Sigma_{n} \cdot(O A)^{n}-\Sigma_{n-1} \cdot(O A)^{n-1} \ldots . . \pm O C^{n},
\end{aligned}
$$

where $\Sigma_{n}$ is the product of all the coefficients of direction for $O P_{1}, O P_{2}, \& c, \Sigma_{n-1}$ the sum of $\wedge$ (the product sq? P.G. Tait) these coefficients taken $\overline{n-1}$ together \& so on. But these coefficients are also the roots of the Equation $x^{n}-1=0$. Now the product of the roots of this Equation with their signs changed is $-1 \& \Sigma_{n}$ is $=$ the product with their signs unchanged. Therefore if $n$ be even $\Sigma_{n}=-1$ but if odd +1 , and in either case $\Sigma_{n-1}, \Sigma_{n-2} \& c$ each $=0$. Hence $C P_{1} \cdot C P_{2} \cdot C P_{3} \cdots C P_{n}= \pm(O A)^{n} \pm(O C)^{n}$; the upper signs to be used when $n$ is even, the lower when odd.

Here $C P_{1}, C P_{2} \& c$ consider the lines both as to length and direction, we must $\therefore$ divide the first or multiply the second by the product of all their coefficients of direction. If $n$ be even the several pairs as $C P_{1}, C P_{n-1}$ are evidently of the form $\left(C P_{1}\right) \cdot 1 \frac{\vartheta}{2 \pi}$ and $\left(C P_{n-1}\right) \cdot 1 \frac{-\vartheta}{2 \pi}$ $\therefore C P_{1} \times C P_{n-1}=\left(C P_{1}\right) \times\left(C P_{n-1}\right)$ and this is true for every pair except $C A=(C A) \cdot+1$ $\& C B=(C B) \cdot-1 \therefore\left(C P_{1}\right) \cdot\left(C P_{2}\right) \cdots\left(C P_{n}\right)=\left(-O A_{n}^{n}+O C^{n}\right) \cdot-1=O A^{n}-O C^{n}$

But if $n$ be odd the several pairs remain as before only no $P$ falling on $B,-1$ is not a coefficient of direction $\therefore\left(C P_{1}\right) \cdot\left(C P_{2}\right) \cdot \& \mathrm{c}=O A^{n}-O C^{n}$ as before.

Cor.1. If $C$ be on the opposite side of $O$ from $A$, the other conditions remaining the same $O C$ is negative. If $n$ be even the deduction in the prop. remains unchanged. But if $n$ be odd, $\left(C P_{1}\right) \cdot\left(C P_{2}\right) \cdot \& \mathrm{c}=O A^{n}+O C^{n}$. Here it may be remarked that when lines as $O A$ are in the original direction, since the coefficient of direction in that case is unity it is immaterial whether we write $O A$ or $(O A)$.

Ex. Let $n=3 \& O C=\frac{1}{2}$,
then, $(A C)=\frac{3}{2},\left(C P_{1}\right)=\left(C P_{2}\right)=\frac{\sqrt{3}}{2}$

$$
\therefore(C A) \cdot\left(C P_{1}\right) \cdot\left(C P_{2}\right)=\frac{3}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}=\frac{9}{8}=1+\frac{1}{8}=\overline{1}^{3}+\frac{1}{2}^{3}=O A^{3}+O C^{3} .
$$

Cor.2. If $C$ be in $O A$ produced the reasoning $\&$ result will be the same as in the prop., only, that now $C A \& C B$ being of the same affection -1 is not a divisor of the second member of the Equation, \& ,

$$
\left(C P_{1}\right) \cdot\left(C P_{2}\right) \cdot \& \mathrm{c}=(O C)^{n}-(O A)^{n} .
$$

11. If from $A$ the extremity of the Diameter (Fig 8) [See Fig 6, Lewis] the circumference be divided into $n$ equal parts \& if these several extremities be joined, then

$$
\left(A P_{1}\right) \cdot\left(A P_{2}\right)\left(A P_{n-1}\right)=n C A^{n-1}
$$

As in former prop. $A P_{1}=C P_{1}-C A, A P_{2}=C P_{2}-C A \&$ so on

$$
\begin{aligned}
& \therefore A P_{1} \cdot A P_{2} \cdots A P_{n-1}=\overline{C P_{1}-C A} \cdot \overline{C P_{2}-C A} \& c \text { to } \overline{n-1} \text { factors } \\
& =R^{n-1} \cdot\left\{S_{n-1}-S_{n-2} \ldots \pm S_{1} \pm 1\right\}
\end{aligned}
$$

where $S_{1}, S_{2} \& c$ are the sum, sum of products two $\&$ two, $\& c$ of all the values of $1^{\frac{1}{n}}$ except unity there being no line drawn from $A$ to the circumference in the direction $C A$. $S_{1}, S_{2} \& c$ are $\therefore$ the coefficients of the Equation $\frac{x^{n}-1}{x-1}$ or of $x^{n-1}+x^{n-2}+\ldots \& c$ with the signs changed for the products of odd ascending roots, unchanged for even ones.

If $\therefore \overline{n-1}$ be even $S_{n-1}=+1, S_{n-2}=-1, \&$ so on,
if $\overline{n-1}$ be odd $S_{n-1}=-1, S_{n-2}=+1 \&$ so on.
$\therefore A P_{1} \cdot A P_{2} \cdot \& c=R^{n-1} \times \pm\{1+1+1$ to $n$ terms $\}= \pm n R^{n-1}$ according as $\overline{n-1}$ is even or odd.

If $\overline{n-1}$ be even, $A P_{1} \cdot A P_{2} \cdot \& \mathrm{c}=\left(A P_{1}\right)\left(A P_{2}\right) \& \mathrm{c} \cdot$ the several pairs of coefficients giving unity for their products.

If $\overline{n-1}$ be odd, then the several pairs give as before their product unity but there remains the factor $A B$ which has for its coefficient -1 .
$\therefore$ in either case $\left(A P_{1}\right)\left(A P_{2}\right) \& c\left(A P_{n-1}\right)=n R^{n-1}$
12. If by this method we undertake to prove that the angles at the base of an Isosceles triangle are $=$ eachother we have $(A C)=(B C)($ Fig 5$)$. [See Fig 3, Lewis]

But $A C=(A C) \cdot 1 \frac{A}{2 \pi}=(A C) \cdot[a+\sqrt{-b}]$,
$C B=A D=(A C) \cdot 1^{\frac{-B}{2 \pi}}=(A C) \cdot\left[a^{\prime}+\sqrt{-b^{\prime}}\right]$.
But $A C+C B=A B$.
$\therefore(A C) \cdot\left(a+a^{\prime}+\sqrt{-b}+\sqrt{-b^{\prime}}\right)=A B=$ a positive quantity
consequently the sinal parts destroy one another or $\sqrt{-b}=-\sqrt{-b^{\prime}}$ or $b=-b^{\prime}$. Therefore the angles $A \& B$ have their sines of equal length but of different affections. The angles themselves $\therefore$ being together less than $\pi$ are geometrically equal to each other.

Cor. Much in the same way we might prove that in every triangle the greater side has the greater angle opposite to it \& vice versâ that the greater angle has the greater side opposite to it.
P.G. Tait.

## Appendices

## A Images to accompany the text: Figures 1-8

Please note that my Figure 3 contains Figures 3, 4 and 5 as referred to in the text, which throws out the subsequent correspondence between Tait's numbering of figures and mine. I feel this is a necessary inconvenience as it allows the reader to view the figures in context. Readers are directed to the appropriate figure by comments within square brackets, e.g. [See Fig 3, Lewis].

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1. $\sqrt{-1}$ is cida infogide or imaginaty " 20 oracinary lle Iechaic puacaty ofrid muat be e ther + ono- can yine whem squater a sogative recult. Emeasing homemes de
 thae the aquenption that enery hire muet be oithur onin incruictand arth the fopililety g-brecing a line in any diveation. $1 \times a$ mianw a line whole longcto in $a$ bem in me divection, $-1 \times a$ man the samolengetopling hits drowe in a defferent divection ans to bay that a line toke length of a cannot he dracon in any other diseress then ore of thexe is absurd: $\sqrt{-1} \therefore$ is not impositlen more thau - $r+l$ ass clow only the derection of the hine tolich it is affico
2. If from $C$ we dras any n under oflinas ruat that thay Chall lecitentinuces pooportion and me be at che oleme time

 caing $C A=, C A_{1}=a, \operatorname{lot}_{2} a^{2}$ on the lincs are in thais cevies $a^{0}, a^{\prime}, \alpha^{2}, a^{2}-$ while the anglu whict the katee witt the line Citf dre $0, V, 29,3 \Omega \mathrm{~V}=$ bicinthe


Figure 1: An extract from Terrot's lecture; Tait's drawing of Figure 1 (Tait-Maxwell School-book)

3. If we now afecme the several lives $C A, b A, b A_{2} \mathbb{N}$ all equal or marie a circle the case vile not he alters. Let $n$ be avivisor of $2 r \pi$ or let $\Omega=\frac{2 r \pi}{n}$. Then the Raxics $a^{n}=a^{\frac{2}{v}}$ is the same in length $Y$ fercition ans GA.: $a^{\prime}=1^{\frac{1}{n}}=1 \frac{9}{2 \pi \pi}$ hoe know from roinary Algebraical principles that the several int roots privity nay he exc-


We: : cole 1 रशता the coefficient of direction because it ropers only to the direction, never to the length gre line. Thus. $a \times \frac{1+\sqrt{-3}}{2}$ i a line $=$ a simply.
4. Let nos newt appose $n=2$, the will he a deainter i $_{\dot{j}} C_{A}=1, b B=-1$. But $a^{2}=1, i a= \pm 1$. But the radii binning $a, a^{2}$, a muct evidently he $=-1, \forall a^{2}=+1$, Troat lat $n=4$, bis, $b D, b B 1, b B$ are the 4 rotor the equation $a^{4}-1=0$. But these roots are $\pm 14$ $\pm \sqrt{-1}$ Here cit + Con are numblipe by $+1 t-1$ seopoctivel $\therefore 60 \times 66$ must he opmolizo ky $+\sqrt{-1}+\sqrt{-1}$ selpectinaly, it being however quite optional which ins estingfinsa $C$ vecaccount positive or negative either in the horizontal or kespernsiciclas tines.
5. It appears from the foregoing Props that if a


Figure 2: An extract from Terrot's lecture; Tait's drawing of Figure 2 (Tait-Maxwell School-book)
 by noviry from At bl. But it isales. Fig. 3.
Chas that its axtuce trowybrence in
poar mbigh not its destince tradles eracest be the tenne dixit museforn
A1 $13+$ then from 18 to 6. Wher.: ( $16 \times$ A
its coefficant 7 disentin) $=(113 x$ its coog
fisient of dovatin) $+(136 x$ it coefficient of divestion). Therefore als He ding of viry tao livies mating an angla witt each other is =
 cones grollid aviosmict whicn we exy ATB, $+B, b=$, $b$ orthel $+6+613_{1}=+13$,
 anglo triangle deseriber on the raxices THD. Ft mide ve exle ATBthe


$=a \times \frac{1+\sqrt{-1}}{\sqrt{2}}$, But 6 b $=\frac{a}{\sqrt{2}}$
CB being kerforienker to ariginich por
$=$ ition is $=\frac{a}{\sqrt{2}} \times \sqrt{-}\left(B_{1}, 4\right)$
$A+b 13=a \times\left[\frac{1}{\sqrt{2}}+\frac{\sqrt{-1}}{\sqrt{2}}\right]=a$
$=a \times 1+\sqrt{-1}$
$=a \times \frac{1+\sqrt{-1}}{\sqrt{2}}=$ N/3.
 a. $1 \frac{60}{360}=a \cdot 1^{\frac{1}{3}}=a \cdot \frac{1+\sqrt{-3}}{2}$, we $=a$ a bisi length 4 limetion , iont $=$ $a \frac{\sqrt{3}}{2}$. in length 4 nsection jointle, wh $=a \cdot \frac{\sqrt{3} \cdot \sqrt{-1}}{2}=a \cdot \frac{\sqrt{-3}}{2}$.
$\therefore \sqrt{2}+b 13=\frac{a}{2}+a \cdot \frac{\sqrt{-3}}{2}$.
3. Let the triange ( Eig5) he Eqvilaterne q et AB $=a \cdot \frac{1+\sqrt{-3}}{2}=$ AB.
 $=a_{0}\left[1^{\frac{1}{6}}+1^{-\frac{1}{6}}\right]=a \cdot\left[1 \frac{1}{6}+\frac{1}{\left.1^{\frac{1}{6}}\right]}, b 13=a \cdot 1^{-\frac{1}{6}} \therefore \cdot A b+b 13\right.$


Figure 3: An extract from Terrot's lecture; Tait's drawing of Figures 3, 4, 5 (Tait-Maxwell School-book)


Figure 4: An extract from Terrot's lecture; Tait's drawing of Figure 6 (Tait-Maxwell School-book)

$$
\begin{aligned}
& \text { 8. } \quad \operatorname{Lin} \overline{A+B}=\sin A \cdot \cos B+\cos A \cdot \sin B \text {. } \\
& \cos A+B=\cos A \cdot \cos B-\sin A \cdot \sin B \cdot \\
& \text { aLt are } A / 3\left(F_{j} ; y\right)=A, B 8+A, \\
& \operatorname{ead}=7 \text {. } \\
& \text { Then by Prop, } 3, b / B=r \cdot 1^{\frac{A}{B T}} \\
& G_{1}=r \cdot \frac{1 \cdot B}{2 \pi}, b_{1} g_{2}=r_{1}, \frac{A+B}{2 \pi} \\
& \therefore G D_{2}=r \cdot 1 \frac{A}{2 \pi} \cdot 1 \frac{\pi}{2 \pi} \text {; } \\
& \text { But by Pout } y, \frac{A}{2 \pi}=\cos A+\sqrt{-1} \text { is } A \text {, } \\
& \rho^{\frac{B}{2 \pi}}=\cos B+\sqrt{-1} \ln B \text {, } \\
& \therefore r^{\frac{A+B}{2 \pi}}=\cos A \times \cos B-\sin A \times \sin B+\sqrt{-1} \text { (H) } \\
& (\sin A \cdot \cos B+\cos A \cdot \sin B) \text {, } \\
& \text { but } \frac{A+B}{2 \pi}=\operatorname{Cos} A+B+\sqrt{-1} \text { din } \frac{A+B}{} \text {. } \\
& \text { Equating then the final or ersival parts of there we have } \\
& \cos A \cdot \cos B-\sin A \cdot \sin B=\cos \overline{A+B} \\
& \sin A \cdot \cos B+\cos A \cdot \sin B=\sin \overline{A+B}
\end{aligned}
$$

Figure 5: An extract from Terrot's lecture; Tait's drawing of Figure 7 (Tait-Maxwell School-book)

10 eotes' Moputies rithe Cirele.
Sx th anienupproma be dividos into on oqual fersts añ foici O1, Of, OP PV (xig. 8.) and aleso jim $I_{1}, P_{2} P_{1}$ art $C_{\text {a }} C$ any ferint in the Diamater. Then
$C P_{1}=O P_{1}-o b,-b P_{2}=O P_{2}-O b K_{A}$.
$\therefore c P_{1} ; b P_{2} ; b P_{3} \cdots \cdot G P_{n}=\Sigma_{n} \cdot\left(O A A_{n}\right.$
$-\Sigma_{n-1}(a t)^{n-1} \cdots \cdots$ 士 $O \mathbb{C} n_{\text {where }}$
$\Sigma_{n}$ is the prosicet of ale the eomeficionts Sountion $x^{n}-1=0$. Euction witt thein hipss the prouct with thieir lijns unchanges. Therefore if a he wn $\Sigma_{n}=-1$ but if o8D $=+1$, and civ itthe care $\Sigma_{n-3}$, $\Sigma_{n-2}$ we enk $=0$. Obence $\left.\left.G P \cdot G P_{2} \cdot G P_{3} \cdot G P_{n}= \pm(0 . A)^{2}+0\right)^{2}\right)$, the uptper kijisa to be vesed when mis ever, the lovarn when ass? Here $b, f_{2}$ ve cmisen the line both as totongth ani discetion, aie muct:: divise the first or madepley the ceem by the prodict of all their coefficient of de. $=$ raction. At on he even the leverel bios as cplope as
 $(G, P) \times\left(B P_{n-1}\right)$ an thi is true for wary kaic easipt $b t-(b A)+1$

 mo fallemin on $B,-1$ is not a erefficiciat of direction $\therefore\left(G P_{1}\right) \cdot\left(G P_{2}\right): v^{2}=O A^{n}-O b^{n}$ as hefore.

Figure 6: An extract from Terrot's lecture; Tait's drawing of Figure 8 (Tait-Maxwell School-book)

## B Editorial corrections

The following table records the necessary editorial corrections made to the transcription:

| Reference | Editorial correction |
| :---: | :---: |
| §2, pg 1 | Tait has ' $\angle A C A_{1}=A_{1} C A_{2}^{2}=A_{2} C A_{3} \& c$ then calling $C A=1, C A_{1}=a, C A_{2} a^{2}$. I cannot see why Tait has the superscript 2 in ' $C A_{2}^{2}$ ' so I have ommited it. I have also added in an equals sign between ' $C A_{2}$ ' and ' $a^{2}$ '. |
| Ex.1, pg 2 | Tait has $\because A C+C B=a \times\left[\frac{1}{\sqrt{2}}+\frac{\sqrt{-1}}{\sqrt{2}}\right]=a=a \times$ $\frac{1+\sqrt{-1}}{\sqrt{2}}=A B .^{\prime}$ I have ommited the ' $=a$ ' as I believe it appears only since there is a break in the line. |
| §7, pg 3 | Tait has ${ }^{\prime} \therefore(\cos \vartheta+\sqrt{-1} \cdot \sin \vartheta)^{p}=\cos p \vartheta+\sqrt{+1} \sin p \vartheta \prime$ which is incorrect: there should of course be a -1 under the second square root sign, rather than +1 . The ink on the original appears smudged here. Perhaps Tait attempted to correct his error. |
| §9, pg 5 | Tait has 'which is $\therefore=r^{2}=\left(C A^{2}\right)=\left(C A_{1}\right)^{2}$. I have repositioned the superscipt 2 to sit in its proper place, outside the bracket '( $C A$ )'. |
| §10, pg 5 | Tait has 'and this is true for every pair except $C A=$ $(C A) \cdot+1 \& C B=(C B) \cdot-1 \therefore\left(C P_{1}\right) \cdot\left(C P_{2}\right) \cdots C P_{n}=$ $\left(-O A_{n}^{n}+O C^{n}\right) \cdot-1=O A^{n}-O C^{n}$, I have added in the bracket around $C P_{n}$ which Tait has forgotten. |
| §11, pg 6 | Tait has 'If $\overline{n-1}$ be even, $A P_{1} \cdot A P_{2} \cdot \& \mathrm{c}=\left(A P_{1}\right)\left(A P_{2}\right)$ $\& c$ the several pairs of coefficients giving unity for their products.' I have added in • on the right hand side of the equation (as a sign of multiplication), as without it, Terrot's meaning is at first unclear. |

Table 1: Editorial changes made to Tait's notes.

