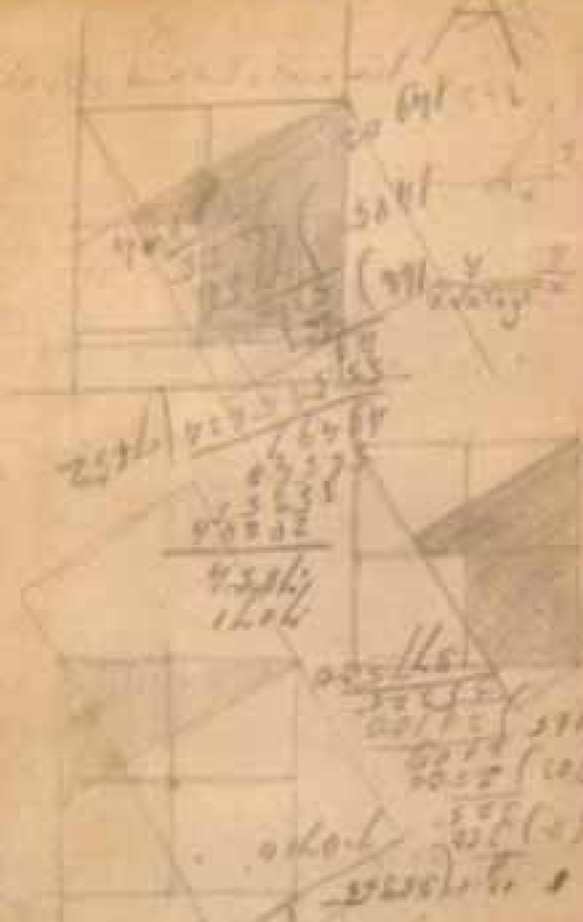
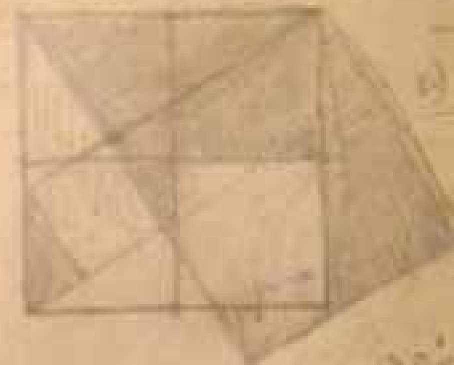


$$\frac{x\,\delta x + y\,\delta y}{\sqrt{x^2 + y^2}}$$



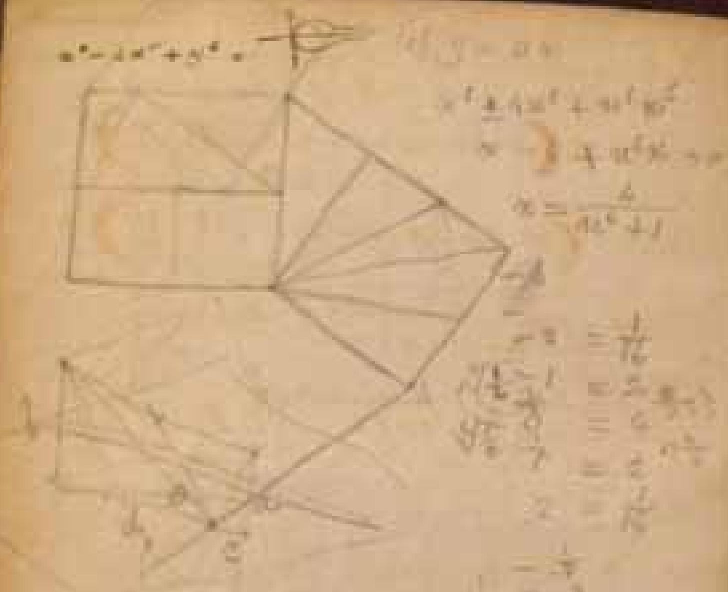
Peter Guthrie Tait

1840

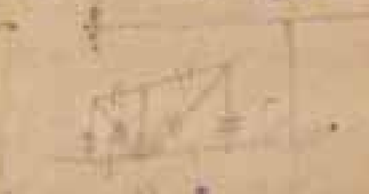
$$y^2 = m^2x^4 + m^2y^2x^2$$

$$y = \frac{mx^2}{\sqrt{1 - m^2x^2}}$$

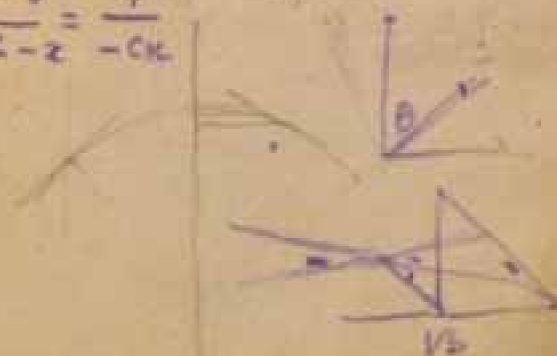
$x^2 - 2x^2 + y^2 =$



$y = x^{\sec \frac{dy}{dx}} = \frac{y}{[illegible]}$



$\frac{Y - y}{X - x} = \frac{y^2}{-cx}$



Let t = length of a tube = bulb + stem [illegible] B = 30 inches. a height in thermometer [illegible] when Barom. falls to B inches hypsometer will be at

$\frac{B+t}{2} \pm \left((B-b)t - (B+b)a + a^2 + \frac{(B+t)^2}{4}\right)^{\frac{1}{2}}$ and when it rises to b inches the [illegible] will be at

$\frac{B+t}{2} - \left((B-b)t - (B+b)a + a^2 + \frac{(B+t)^2}{4}\right)^{\frac{1}{2}}.$

To draw a tangent at any point A to a curve which seen from B covers every object.

Join AB, Draw BC perpendicular to AB, [illegible] meeting [illegible] in C, Draw CD // AB cutting [illegible] in D. Join DA [illegible] is the tangent required.

J. C. Maxwell. Sept

Equation to the tangent when this x, y, AB = c, is

$Y = \frac{c^2 X - cxy}{cx - [illegible]}$

If tan u = x, & tang u' = (x + h),

$u' = u + \cos u \cos u \frac{h}{1} - \sin 2u \cdot \cos^2 u \frac{h^2}{2} - \cos 3u \cdot \cos^3 u \frac{h^3}{3} + \&c$

$D(\sqrt{x^2 \pm y^2}) = \frac{x\,dx \pm y\,dy}{\sqrt{x^2 \pm y^2}}$ see previous page.

$yy' = 2m(x + x')$

$yy' = 2a[illegible](x + x')$

$y = 2a[illegible]\frac{x'}{y'}$

$$x = \frac{1}{2}\left\{-\epsilon^{y} - \epsilon^{y}\right\} + C = C - \frac{1}{2}\left\{\epsilon^{y} + \frac{1}{\epsilon^{y}}\right\}$$

$$(\epsilon^{2y} - 1)\left(\frac{dy}{dx}\right)^2 = -2\epsilon^{y}\frac{dy}{dx}$$

$$\left\{1 - \epsilon^{2y}\right\}\frac{dy}{dx} = 2\epsilon^{y} \quad \therefore x = \frac{1}{2}\int(\epsilon^{-y} - \epsilon^{y})\,dy$$

$$\epsilon^{y}dy = \sqrt{dy^2 + dx^2} + dx$$

$$\epsilon^{2y}dy^2 - 2\epsilon^{y}dy\,dx + dx^2 = dy^2 + dx^2$$

$$\therefore \frac{dx}{dy} = \sqrt{1+p^2} \quad \therefore y = \int\left(\sqrt{1+p^2} + p\right)$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$y = \log x + \text{constant}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\frac{dx}{dy} = \int\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

equation is $y = \frac{mx^2}{\sqrt{1 + m^2x^2}}$

$$\frac{y}{(x^2+y^2)\sqrt{x^2+y^2}} = m \qquad \sin\theta = \frac{r^2}{a^2}$$

$$y^2 = m^2x^6 +$$

$$y^2 = m^2(x^2+y^2)^3$$

$$x = \int\sqrt{dx^2 + dy^2} \qquad \therefore \sin\theta = m r^2$$

All time, balanced by gravity ... curve ABCA.

$$x = C - \frac{1}{2}\left(\epsilon^{y} + \frac{1}{\epsilon^{y}}\right)$$

$$\frac{y}{\sqrt{y^2 + x^2}} = \frac{1}{y^2 + x^2}$$

$$\therefore \frac{y}{m} = \left\{y^2 + x^2\right\}^{\frac{3}{2}}$$

$$\therefore y^2 = C\left\{y^2 + x^2\right\}^3$$

A row of pebbles on the shore probably constituted the first method of numeration. The number 15 would be thus represented

. . . .

Afterwards a better method that of laying them in pairs was adopted. The number 13 would be represented by it in the following way

. . . .
. . . . . .

Then the method of increasing the pebbles in size was used. 15 would then be thus represented if the pebbles were 3 times the size

o o o o o

or of 5 times the size thus

0 0 0

Scales Then another way was found out, that of Regulating their value by their position forming them into a scale increasing by two or any other number according to their position in the different columns as under

Binary

32	16	8	4	2	1
.	.	.	.	.	.

Palpable Arith

Quinary

3125. 625. 125. 25. 5. 1.

Denary

100000. 10000. 1000. 100. 10. 1.

For the Binary Scale two written characters will be sufficient But the Denary scale is now in general use and the Arabic numerals are used for this purpose.

Trans. fr. one scale to another

To trans fr. one scale to another it is only necessary to divide or to multiply by the index of the scale &c

If 111100 of the Binary Scale is to be reduced to the Denary each figure must be multiplied according to its position that is as often as there are figures between it & the unit column itself included by the index of the scale & the products added together thus

111100

0 = 0

2 x 0 = 0

2 x 2 x 1 = 4

2 x 2 x 2 x 1 = 8

2 x 2 x 2 x 2 x 1 = 16

2 x 2 x 2 x 2 x 2 x 1 = 32

60

(2) 111100
12 | 1110
6 | 0, 6 = 60

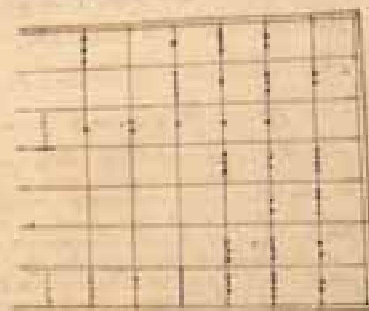
Palpable Arith.

And the 60 may be reduced to the Binary scale by division by 2 all its reduced to the lowest results as follows

2|60
2|30 – 0
2|15 – 0
2|7 – 1
2|3 – 1
2|1 – 1
1 – 1

The Roman Abacus

Abacus



By the above will be seen the rat[ionale] of carrying & borrowing in Common Arithmetic The first column amounting to 12 is equal to 1 Ball placed on the second column & 2 on the first. The second amounting to 14 is equal to 1 on the third & four on the second &c

Or, to take another example
If from the number 53
be taken 48
there remains 5

Palpable Arith.

we have here 8 units of the first order to be taken from three of the same order increased by one unit of the second order taken from the five on that line, thirteen leaving five, and four units of the second to be taken from the 4 still remaining on that column leaving 0.

Roman numerals

The introduction of the letters I V X L C D & M probably originated in the crossing of the figure representing ten to show that the Denary number had been arrived at and this cross X was then divided into two parts to represent the intermediate number five V. On arriving at the second Denary number two marks would be added thus ⊏ which being divided as before will give the intermediate number 50 - L. And the third denary number would have three strokes thus M which being halved gave rise to the character D expressing 500

Napiers Rods

For facilitating the processes of multiplication & Division Baron Napier of Merchiston near Edinburgh invented the Rods called after his name which are constructed by arranging the Multiplication Table in the manner

Palpable Arith.

shown in the fi

To Multiply by the assistance of Napier's Rods. Tabulate the Rods that is arrange them so that the top column shall express the numbers to be multiplied. Then in the left hand column the figures of the Multiplier are to be taken beginning with the unit figure first & the sum opposite on the Rods is to be read off beginning from the right hand. Where two numbers are in one rhomboid they are to be added together & if the product exceeds 10, 1 is to be carried to the next Rhomboid. The sum is then to be written down & the 2nd figure of the multiplier is to be proceeded with in the same manner observing that the first figure shall occupy

the tens place & so on with the rest each
being placed in its respective order
ie tens under tens hundreds under
hundreds &c. Afterwards they are to
be added together.

In division the sum to be divided
is to be written on a piece of paper and
the Divisor tabulated. The divisor near-
est to the Dividend is to be sought on
the Rods and the number on the index
to be taken as the Quotient. The prod-
uct is to be subtracted as usual. The
next figure of the dividend is to be
brought down & the Quotient will
be found in the same manner &
so on until the operation is comp-
leted.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Satellites of Jupiter.

	[illegible]	Mean Sy. [illegible]	[illegible]
1	266,490.	1ᵈ 18' 28."	1 · 18 · 27 · 33.
2	420,815.	3ᵈ 13' 14."	3 · 13 · 13 · 42.
3	671,234.	7ᵈ 3' 43."	7 · 3 · 42 · 33.
4	1180,582.	16ᵈ 16' 32."	16 · 16 · 31 · 50.

The Inclination of the [illegible] to the plane of the [illegible] nearly = 89° 43' 15" or 91° 6' 40".

[illegible]

As the [illegible] of the [illegible] [illegible]
[illegible] [illegible] [illegible] [illegible] [illegible]
it is evident that if the solid formed by
the revolution of any [illegible] [illegible]
[illegible] [illegible] [illegible] [illegible]
would converge to the other [illegible] the rays from one focus
[illegible] [illegible] [illegible] [illegible] [illegible]
[illegible] [illegible] altered by a [illegible]
centre of the greater [illegible] a sphere whose
rays falling [illegible] to the [illegible] because the
sphere maintain their original direction

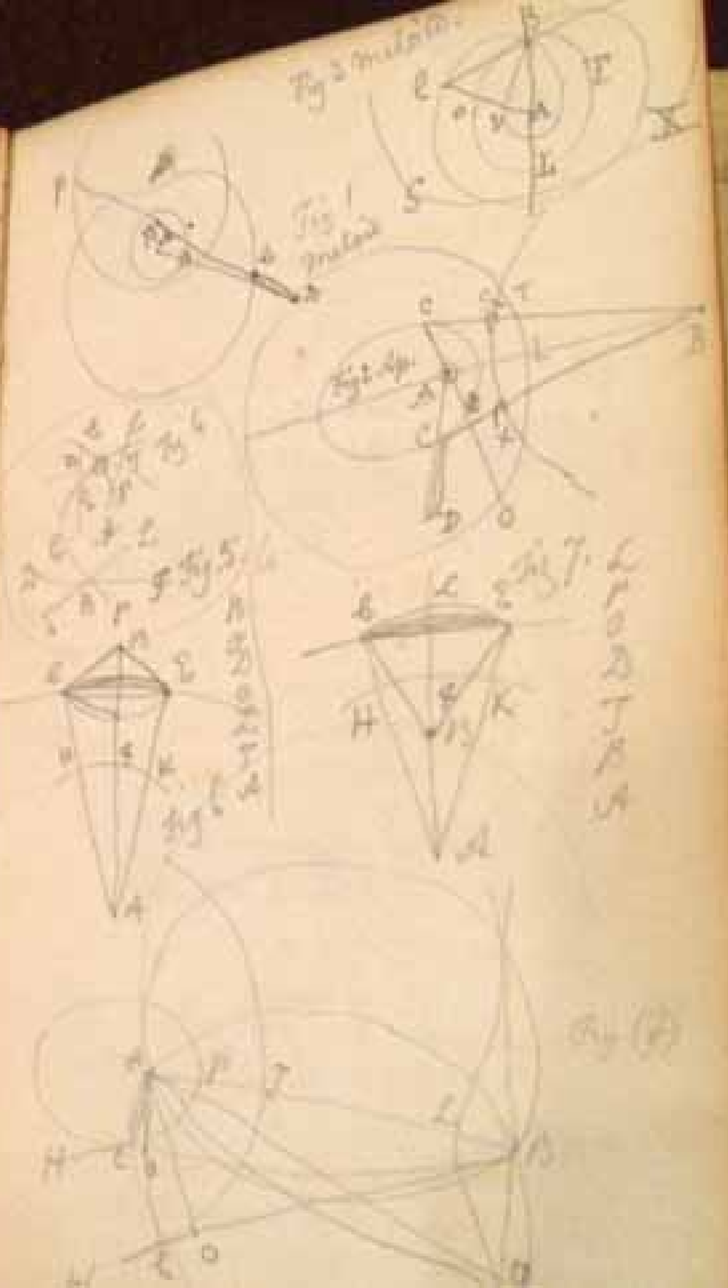
The ellipse & Hyperbola fig [illegible]
cause [illegible] rays to converge [illegible] Prop III they
are ovals one of whose foci is at an
infinite distance.

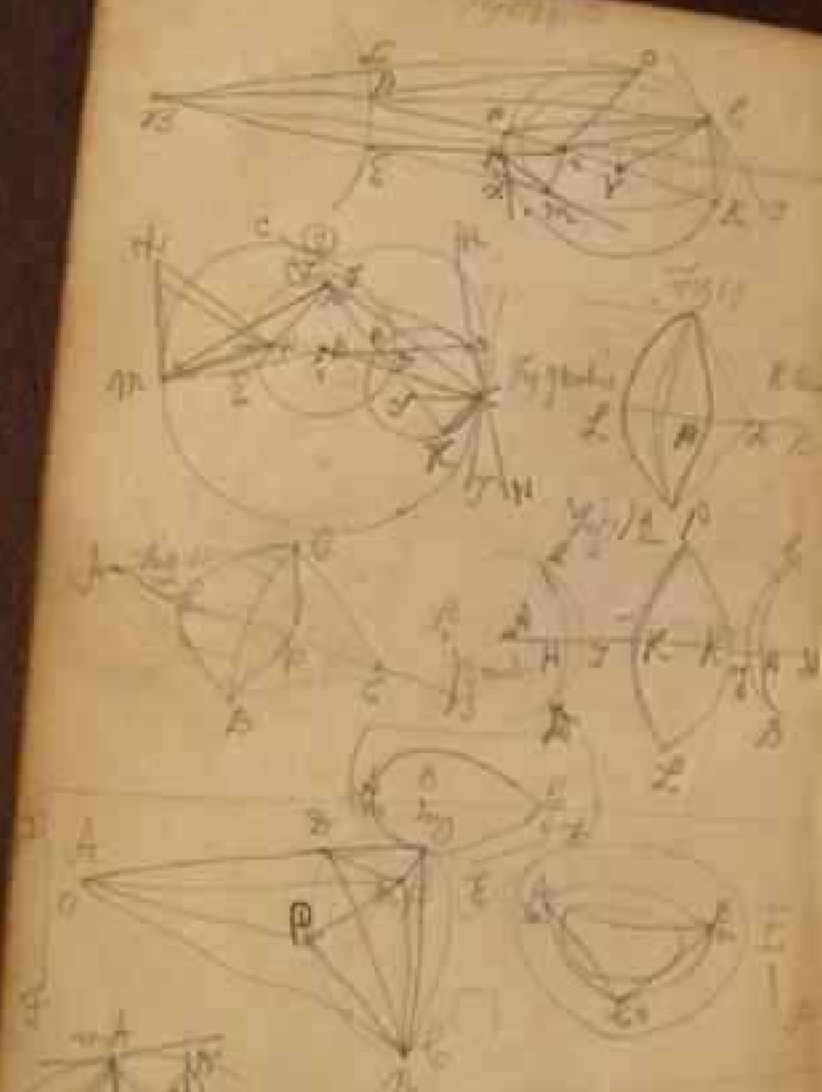
Fig 4 is a combination of two
Hyp[illegible] and rays from A are refracted
to B.

Fig 10 [illegible] is an Ellipse converging rays to
B. [illegible] is a [illegible], foci A, B converging the rays to
A.

Fig 11 [illegible] is an oval [illegible] foci A, B converging rays
from A to B. [illegible] is a [illegible], foci C, B refracting
the rays to C.

Fig 12 [illegible] is a circle [illegible] rays from
A, [illegible] is a [illegible] in Prop 8 [illegible] [illegible]
the rays as if they had come from B. [illegible]
[illegible] of Hyperbolas reflecting from B & B. the whole
[illegible] [illegible] [illegible] [illegible]

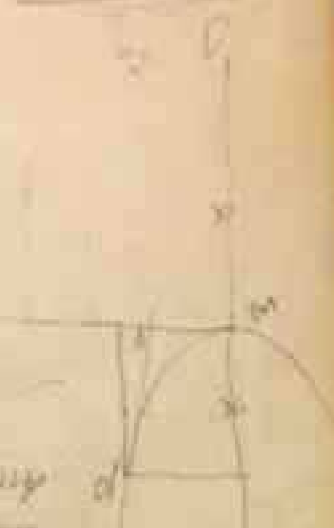


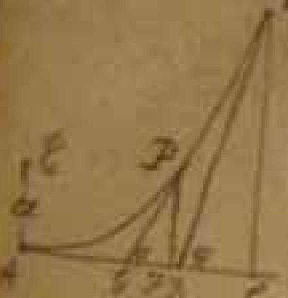


$$\frac{c(x+z)}{c^{w}z} = cnp = xp\,\frac{x(x+z)}{x^{z}+xz}$$

$$\frac{1}{(2n-1)^2}$$

$$\ldots \int w = 1$$





[illegible] the equation [illegible]
[illegible]
[illegible]

$$\therefore \frac{dy}{dy} = w + dw$$

$$w\frac{dy}{dx} + \frac{dy}{dx}dw = 1$$

$$w + dw = \frac{dx}{dy}$$

$$dw = \frac{dx}{dy} - w$$

$$dw + d^2w = \frac{d^2x\,dy - d^2y\,dx}{dy^2}$$

[illegible]

$$\therefore \int \frac{1}{(a+x)^2} = \frac{dy}{dx}$$

$$\int \frac{dx}{[illegible]} = \frac{dy}{dx} =$$

$$-\frac{1}{x+a} \quad \therefore$$

$$dy = -\frac{dx}{x+a}$$

$$-\frac{dx}{z} = [illegible]$$

$$y = -\log(x+a)$$

$$\frac{dy}{dx} = \frac{1}{x+a} = -\int \frac{dx}{(x+a)^2}$$

[illegible]

$$\frac{dy}{dx} \propto \frac{\cos\varphi}{\sin\varphi}$$

$$\propto \frac{\cos^2\varphi}{\sin\varphi}\int_x f(x)$$

$$\frac{dy}{dx} = m\frac{\cos^2\varphi}{\sin\varphi}\int_x f(x)$$

$$1 = m\frac{dx}{\sqrt{dx^2+dy^2}}\int_x f(x)$$

$$\sqrt{dx^2+dy^2} = m\,dx\int_x f(x).$$

$$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = m\int_x f(x)$$

$$\frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = m f(x)\,dx$$

$$f(x) = \frac{\frac{dy}{dx^2}\frac{d^2y}{dx^2}}{m\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$n(dy^2+dx^2) = \frac{dy^2 d^2x^2 - 2\,dx\,dy\,d^2x\,d^2y}{dy^2}$$

$$n\left(dy^4 + \frac{dy^2dx^2}{n-1}\right) =$$

$$dp^2 = n^2dy^4 + n^2dy^2p^2$$

$$(dy)^4 + p^2(dy)^2 + \frac{p^4}{4} = \frac{dp^2}{n^2} + \frac{p^4}{4} = \frac{4p^2+n^2p^4}{4n^2}$$

$$(dy)^2 =$$

$$\frac{dx}{dy} = \alpha\int f(x)$$

$$y = \varphi(x)$$

$$\omega \propto \frac{\sqrt{dx^2+dy^2}}{dy}$$

$$\omega = m\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

MENSURATION
or
HEIGHTS & DISTANCES

$$\frac{dx}{dy} = n\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$\frac{d^2x}{dy^2} = n\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)^2 = n^2 + \left(n\frac{dx}{dy}\right)^2$$

$$\frac{d^2x^2}{dy^4} = n^2 + n^2\frac{dx^2}{dy^2}$$

$$d^2x^2 = n^2dy^4 + n^2dy^2dx^2$$

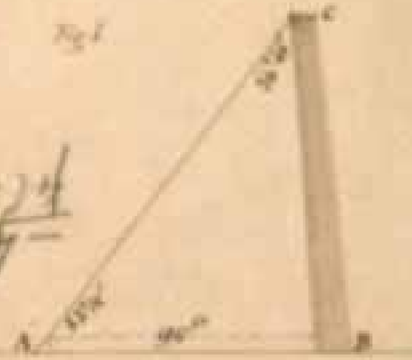
$$\frac{dy\,d^2x - dx\,d^2y}{dy^2} = n\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$(d(dx))^2 = n^2(dy)^4 + n^2(dy)^2$$

PROBLEM I.

To find the height of an accessible object.

Fig. 1



At a station 90 feet from the foot of a pillar CB the angle of elevation CAB was found to be 53°16′ the height of the eye being 5 feet what was the height of the pillar

As Sine ACB 36°44′	=	9.776768
Is to Sine CAB 53°16′	=	9.903864
So is Logarithm AB 90	=	1.954243
to Logarithm CB 120·5		11.858107
Add height of eye 5.		9.776768
125·5		2.081339

PROBLEM II

To find the height of an inaccessible object

Fig II

The angle of elevation ADB of a hill AB taken at the bottom D was 40° and 300 feet farther off at C the Angle was 30°15' what was its height

1 To find AD

As Sine CAD 9°45'	9 - 228784
is to Sine ACD 30°15	9 - 702236
So Logarithm of 300	2 - 477121
	12 - 179357
	9 - 228784
is to Logarithm of 892·4	2 - 950573

2d To find AB

As Radius Right Angle DBA 10-	
is to Sine Angle ADB 40°	9·808067
So Logarithm AD 892·4	2·950573
to Logarithm AB 573·7	2·758640
add height of eye 5	

PROBLEM III

To find the distance of an inaccessible object

Fig III

Let the Angle CAB be 63°45', CBA 82°16', and the base AB 1200 links what is the distance CA

As sine angle ACB 35°59'	9.747374
: Sine angle CBA 82°16'	9.996032
∴ Logarithm AB 1200	3.079181
	13.075213
	9.747374
: Logarithm CA 2127.3	3.327839

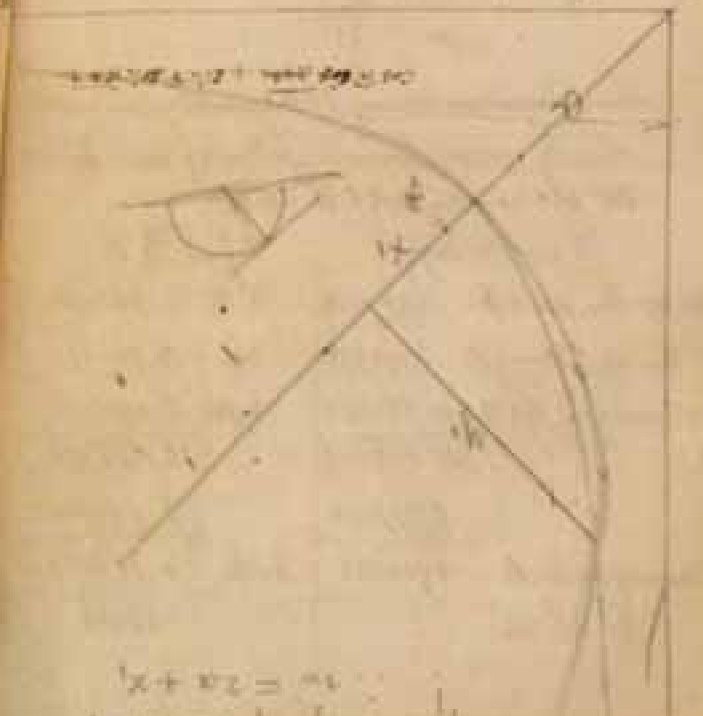
PROBLEM IV

To find the distance between two inaccessible objects

Fig IV

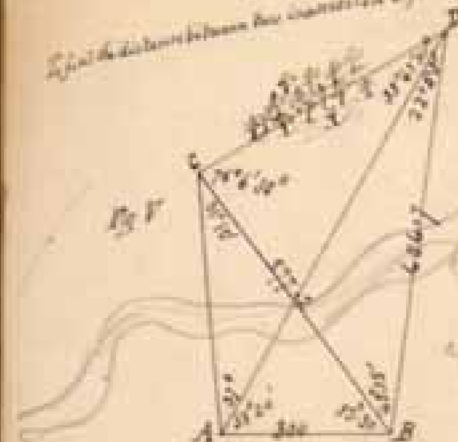
Find the distance between A & B separated from each other by a marsh from the following data AC 500 BC 450 and angle ACB 66-30

To find the Angles		To find AB	
Ar. Logarithm 450+500	2.977724	S, B	9.963250
Logarithm 500−450	1.698970	S, C	9.962391
the tangent of half A+B	10.183342	L, AC	2.698970
	11.862312		12.661361
	2.977724		9.963250
Cotangent of half B−A 4°35′38″	8.904588	L, AB	2.718155
			522.6



PROBLEM V

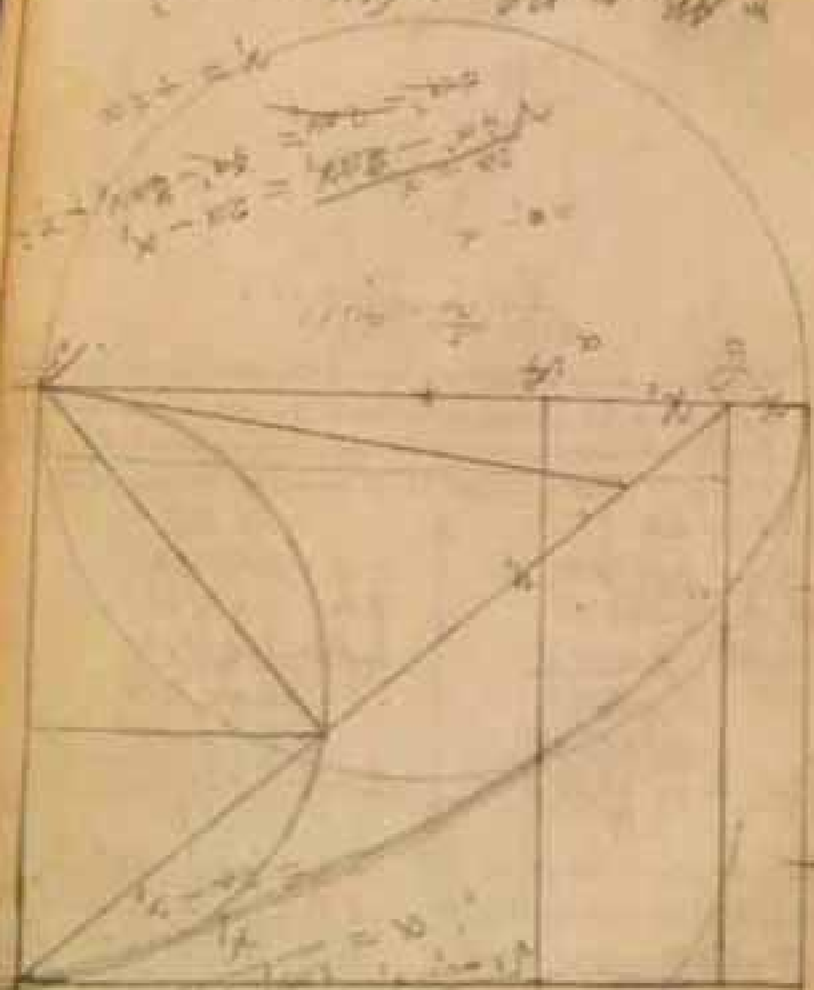
To find the distance between two inaccessible objects



Wishing to know the distance between a house D & a Church C both on the further side of a river and separated from one another by a wood I chose two stations A and B 300 yards apart and found the angles as follows ABC 35°30' CBD 48°15' CAD 57° & DAB 95°20' what is the distance CD

To find CB		To find DB	
S, C	9.713935	S, D	9.590387
CS, A	9.998116	S, A	9.929989
L, AB	2.477121	L, AB	2.477121
	12.475237		12.407110
	9.713935		9.590387
L, 577.2	2.761302	L, 656.7	2.816723

To find CD		To find CD	
	3.091315	S, C	9.987092
L, 656–577.2	1.897627	S, B	9.851372
T, ½ C+D	10.379926	L, BD	2.816723
T, ½ C–D	12.277851		12.668095
8°44'	3.091315		9.987092
	9.186236		



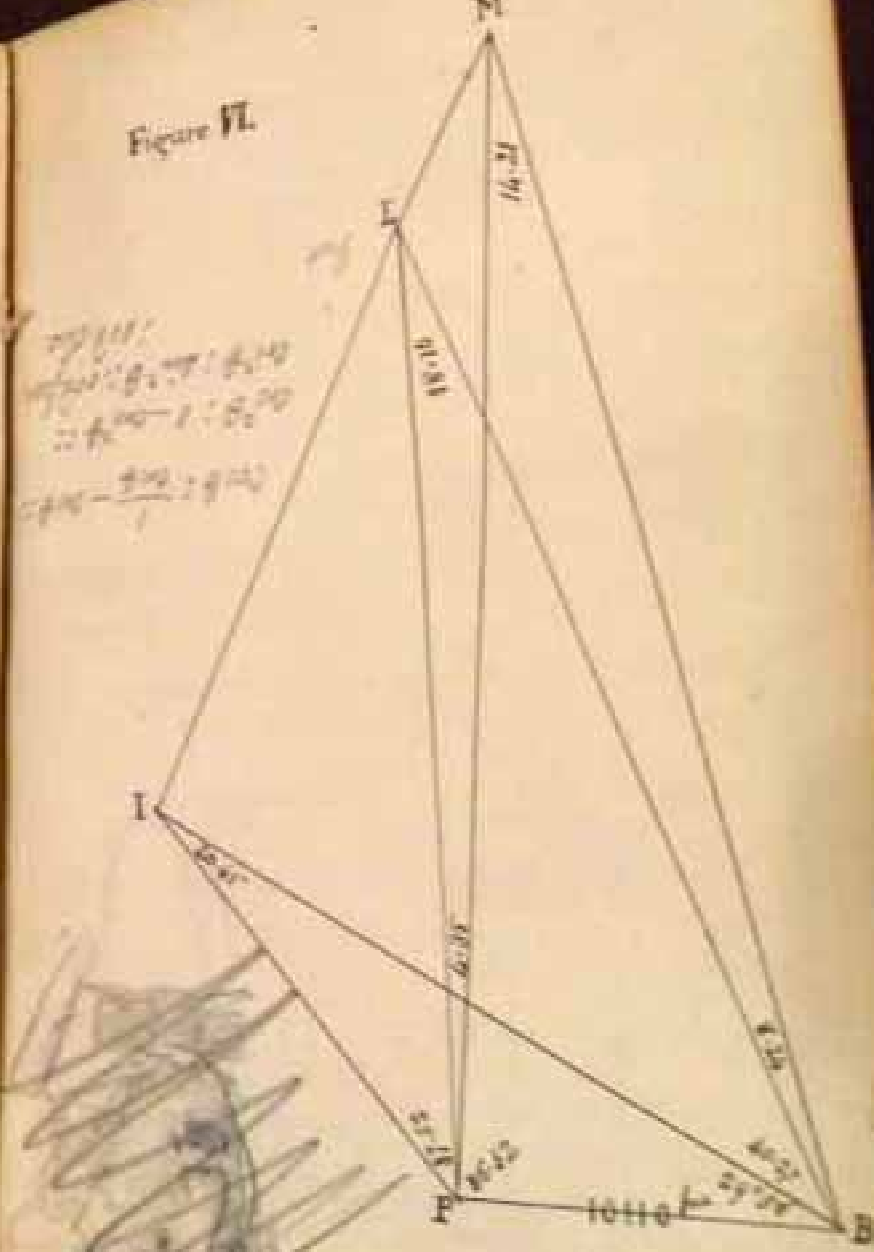


Figure VI.

[illegible]

[illegible]

Let M (fig 6) Represent Nelsons Monument [illegible] the [illegible] I the [illegible] of [illegible] P the [illegible] at [illegible] also let PB = [illegible] Find [illegible] PBI 24° 51' IBL 40° 27' ∠BM 8° 24' MPB 86° 52' ∠BM 46° 35' and IPL 37° 52' Find all the distances not given

1 To find IP

S, I	9.549360
S, B	9.698794
L. 10110	4.004751
	13.702845
	9.549360
L.	4.153485

2 To find IB

S, I	9.549360
Cos, P	9.888548
L. 10110	4.004751
	13.893299
	9.549360
L.	4.343939

To find PM

S, M,	9.394179
S, B	9.491624
L. 10110	4.004751
	13.496375
	9.394174
L.	4.602196

To find BM

S, M,	9.394179
S, P,	9.999350
L. 10110	4.004751
	14.004101
	9.394179
L.	4.609922

To find IB   To find BL

S.L.	9·493851	S.L.	9·493851
S.B	9·974032	COS.P	9·499861
L10110	4·004751	L10110	4·004751
	13·478783		14·004612
	9·493851		9·493851
L	4·484932	L.	4·510761

To find I L

:L,IB+BL	4·736317
:L,BL−BI	4·014521
:T,½I+L	10·433458
	14·447979
	4·736317
:T,½I−L	9·711662

$\frac{2mx}{r}$ $\frac{2mx}{\sqrt{m^2+x^2}}$

$y = \frac{2mx}{\sqrt{m^2+x^2}}$

[illegible]

v = velocity at [illegible]
C = [illegible] at D

$v^2 = c^2 \frac{[illegible]}{[illegible]}$

[illegible] at any pt P [illegible] at any other R [illegible]

[illegible] $v^2 = c^2$ [illegible]

[illegible] in terms of the Rad. Vector [illegible]

$y = mx + \beta$

$y = \tan 2\alpha \, x - \frac{\tan 2\alpha}{\tan \alpha} \beta$

$y = mx + \beta$

[illegible] $+ \beta = \tan 2\alpha \, x - \frac{\tan 2\alpha}{\tan \alpha} \beta$

[illegible]

$y^2 = \frac{4m^2x^2}{m^2+x^2}$

$x^2y^2 + x^2y^2 = 4m^2x^2$

To free the [illegible] the cubic equation

$x^3 + ax^2 + bx + c = 0$ & let $x = y - \frac{a}{3}$

then, $y^3 - ay^2 + \frac{a^2 y}{3} - \frac{a^3}{27}$
$+ ay^2 - \frac{2a^2 y}{3} + \frac{a^3}{9}$
$+ by - \frac{ab}{3}$
$+ c$

$y^3 + (a - a)y^2 + \left(b - \frac{a^2}{3}\right)y + \frac{2a^3 - 9ab + 27c}{27} = 0$

∴ If the coefficient of y be $= 0$ or $b - \frac{a^2}{3} = 0$, [illegible] $b = a^2$; that is if the square of the coefficient of the second term be = 3 times the coefficient of the third, both of these terms will vanish in the equation.

(P.G. Tait, fecit)

2:2:18.

(P.G. Tait)

4. Theorem

If in any biquadratic equation, denoting the coefficients of the second, third & fourth terms respectively by a, b & c and the [illegible] quantity [illegible] by m, the following equation hold good viz $a = \frac{\sqrt{24b - 48m + 81} + 9}{3}$ then by applying the usual rule for destroying the second term the fourth will also vanish and will leave only the 4th & 2nd powers of x in the transformed equation so that it may be solved as a quadratic.

Let the equation be

$x^4 + ax^3 + bx^2 + cx + d = 0$, & let $\frac{c}{a} = m$,

it will be found on going over the same process as in last theorem, that when the coefficient of x^3 has vanished, that of x will be $\frac{-3a^3 + 4a^2[illegible] - 16}{16}$ equating this with zero & solving we find the relation between the constants must be $a = \frac{\sqrt{24b - 48m + 81} + 9}{3}$

(P.G. Tait, fecit)

2:2:18.

(P.G. [illegible])

Theorem

5\. 

If AH, AO be drawn making any angle [illegible] with the axis AM of [illegible] parabola & [illegible] any double ordinate [illegible] be drawn

$(OH \cdot HF) = (AO \cdot OE)$.

[illegible]

[illegible] $= OH^2 \pm FH \cdot HF$,

But from geometry $OA \cdot AE = OE^2 = OH^2$; $AO \cdot OE + AO^2 = OH^2$;

[illegible] $= OH^2 \pm AO \cdot OE$

∴ $AO \cdot OE = FH \cdot HF$.

2:2:18.

(P.G. Tait)

(P.G. Tait fecit)

Problem

6\. 

Supposing a point to move along BD with $\frac{m}{n}$ times the velocity with which it moves along any line AB, [illegible] AB & BD [illegible] the point C so that $\frac{AC}{n} + \frac{CD}{m}$ may be the least possible.

Let $AB = a$, $BD = b$ & $BC = x$.

then $D\left(\frac{x}{n} + \frac{b - \sqrt{x^2 - a^2}}{m}\right) = 0$, or $\frac{Dx}{n} - \frac{x\,Dx}{m(x^2 - a^2)^{\frac{1}{2}}} = 0$,

or $m^2x^2 - m^2a^2 = n^2x^2$ ∴ $x = \left(\frac{m^2a^2}{m^2 - n^2}\right)^{\frac{1}{2}}$.

2:2:18.

(P.G. Tait)

(P.G. Tait fecit)

Problem

7\. To draw a tangent to an Ellipse passing through a point either on or without the curve.

1\. When O is on the curve.

Describe a circle on the axis Major & draw a tangent to it at the point where the ordinate through O cuts it this will meet the axis Major produced in C, & CO joined is the tangent required.

2\. When O is without the curve.

Draw OM perp. to axis Maj. produced, produce it so that $AM : OM :: Ax. Maj. : Ax. Min.$ from A draw the tangent AB to the circle on the Ax. Maj. draw [illegible] perp. Join OC it is the Tangent required.

[illegible]

(P.G. Tait, fecit)

[illegible]

$= 2 + \frac{\text{[illegible]}}{\text{[illegible]}}$

[illegible] ∴ Any number [illegible] with its reciprocal $= 2 +$ the square of the difference [illegible]

[illegible] the [illegible] (which is [illegible])

2. 3. [illegible]

9. To find the area of a regular 14 sided fig. inscribed in a circle.

$AB^2 = [illegible]$

[illegible] substituting in (1) we have

[illegible]

Reducing we have [illegible] $= 0$ (2)

[illegible] $r = 5 \cdot 044997$ [illegible]

The other roots of (2) are $r = 0 \cdot 6237$ & $r = 0 \cdot 3220$ [illegible]

& when $r = 1$, $a = 0 \cdot 19802$ [illegible]

∴ the area of a regular 14 sided fig. inscribed in a circle [illegible] radius $= 1$ [illegible] $= 2 \cdot 2022$ [illegible]

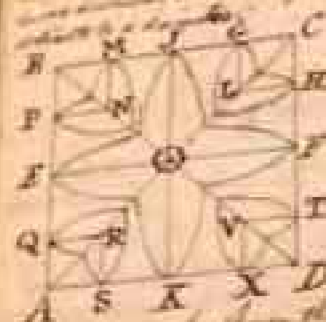
2. 3. [illegible] (P. G. Tait)

10. Problem.

The area of a pentagon inscribed in a circle whose radius is [illegible] will be found $= \frac{5a^2}{8}\sqrt{10 + 2\sqrt{5}}$.

(P. G. Tait)

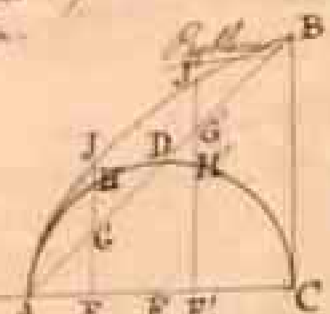
[illegible]



[illegible] of the right cylinder whose axis EF [illegible] whose diameter $SK = EF$ the diameter of the [illegible]

2. 3. 2.

Problem



[illegible] for a semi-parabola [illegible] of AC = parameter. [illegible] AB [illegible] $AF = x$, $AF' = x'$ [illegible]

[illegible] square root of the difference of the squares of the ordinates of the parabola & straight line AB or [illegible] ∴ the solid content of the figure we have just described is $=$ (the [illegible] content of a paraboloid whose height $=$ Parameter [illegible] diameter of [illegible]) $-$ (the solid content of a square pyramid whose base $=$ [illegible] & height $=$ [illegible] of Parabola). Which was to be found.

(P. G. Tait [illegible])

11. Problem

To construct the roots of $y^3 + a^2y - bc^2 = 0$ by means of the Parabola & Hyperbola.

Draw any straight line AH, cut off $AF = \frac{a^2}{b}$ and with FE for the axis, b for the parameter, & E for the vertex describe the parabola EO. At A raise the perpendicular AJ. Cut off AB = c, and complete the square ABCD. With AJ, AH as asymptotes & C a point on the curve describe the rectangular Hyperbola JCOH. The ordinate OE drawn through their point of intersection will be one of the roots of the given equation.

Let OE = y, AE = x.

y [illegible] $\therefore x = \frac{y^2}{b}$ and $\left(\frac{a^2}{b} + \frac{y^2}{b}\right) y = c^2$

or $a^2y + y^3 = bc^2$ or $y = OE$ is a root of the given equation.

Which was to be done.

(P. G. Tait, fecit)

(P. G. Tait)

12. To find the Area of a curve whose equation is $y^m = a^{m-n} x^n$.

$$D(\text{Area}) = a^{\frac{m-n}{m}} x^{\frac{n}{m}} dx$$

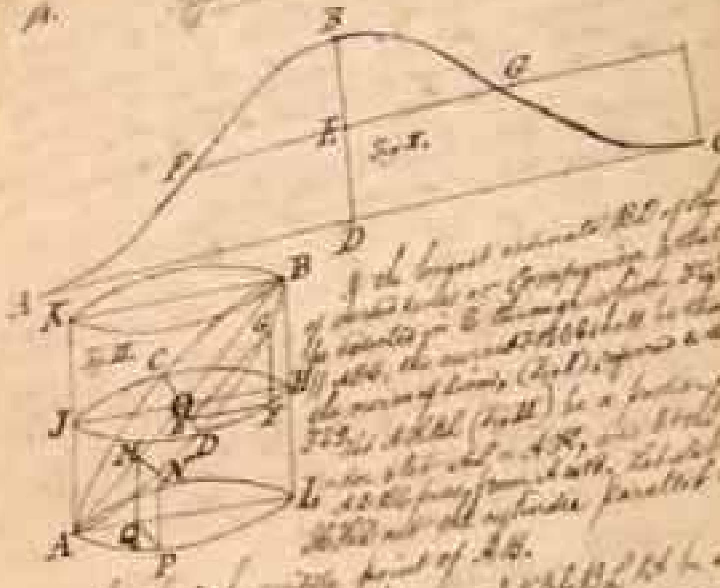
$$\therefore \text{Area} = \frac{m}{m+n} a^{\frac{m-n}{m}} x^{\frac{n}{m}} x = \frac{mxy}{m+n}$$

which was to be found.

(P. G. Tait, fecit)

(P. G. Tait)

11.



If the longest ordinate SD of the curve of sines [illegible] [illegible] the [illegible] ABCD [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] (PKL) be a portion of [illegible] [illegible] [illegible] parallel to the base [illegible] [illegible] cylinder [illegible] [illegible] [illegible] AB.

[illegible] through C the middle point of AB [illegible] [illegible] [illegible] [illegible] [illegible] half of [illegible] [illegible] [illegible] [illegible] on a plane it will [illegible] correspond to [illegible] from the [illegible] [illegible] the companion to the cycloid [illegible] the line HG will correspond [illegible] [illegible] diameter GB any [illegible] [illegible] (Fig. 2). Draw any ordinate [illegible] perpendicular to [illegible] [illegible] then [illegible] the companion of the cylinder [illegible] [illegible] [illegible] will be half the curve of (sine DB) $\times$ [illegible] $\therefore$ [illegible] [illegible] in the same manner be proved half of sine. [illegible] may in the same manner be proved half of [illegible] [illegible] the companion to the Cycloid when [illegible] [illegible] the axis [illegible] through the middle of [illegible] a line drawn parallel to its axis through the middle of the longest ordinate becomes the curve of sines.

Q. E. D.

(P. G. Tait, fecit)

(P. G. Tait)

15.

[illegible]

Let x = [illegible] miles travelled per hour

∴ [illegible] behind A $\frac{10x-15}{3}$ miles.

[illegible]

$\therefore$ [illegible] $: \frac{10x-15}{3} - \frac{8x}{3} + 19 : \frac{8x}{3} - 19.$

$\therefore 30x - 45 - 24x + 171 = \frac{32x^2}{3} - 76x,$

$\therefore 32x^2 - 246x = 378.$

$\therefore x^2 - \frac{123x}{16} + \frac{15129}{1024} = \frac{378 \times 32 + 15129}{1024} = \frac{27225}{1024}$

$\therefore x = \frac{123 \pm 165}{2 \times 16} = 9 \text{ or } -\frac{21}{16}.$

∴ A & B travelled each 9 miles per hour and the distance was $= \frac{10 \times 9 - 15}{3} = \frac{75}{3} = 25$ miles.

Which were to be found.

P. G. Tait

P. G. Tait

16.

Let ABC be a semicircle & let the ordinate [illegible] be divided in D so that [illegible] EF be cut [illegible] the locus of [illegible] ellipse for $EF = (1 - [illegible])$ [illegible] so that every ordinate is a [illegible] quantity) times that of the circle.

P. G. Tait fecit

17.

Let [illegible] the [illegible] in A. Draw the diameter AG ∥ ML. Draw any ordinate IK & in the line LF passing through K ∥ ML take LH, KG each equal to IK. The locus of L & G will be composed of two equal ellipses [illegible] & [illegible] whose centres are in O the centre of the circle & which touch [illegible] in M & P.

The equation to the compound curve calling AQ x & QL y is $y = r \pm \sqrt{\frac{r^2 \pm \sqrt{2r^2x^2 - x^4}}{2}}$

From this we find the longest abscissa AQ $= \sqrt{2} \cdot r$.

The longest ordinate AS $= 2r$.

& the ordinates at the origin, AH & AG $= \frac{\sqrt{2} \pm 1}{\sqrt{2}} r$.

The angles LBK, KBG each $= 45°$, & KBG $= 90°$.

The reduced equation is

$$4y^4 - 16ry^3 + 20r^2y^2 - 8r^3y + r^4 + x^4 - 2r^2x^2 = 0$$

which is the product of

$$2y^2 + 2xy + x^2 - 4ry + 4rx + r^2 \quad \&$$
$$2y^2 - 2xy + x^2 - 4ry - 4rx + r^2.$$

The angle OKA is found by the equation $\tan 2\theta = \frac{-8}{a-c}$ $= \frac{2}{1}$ or $\frac{-2}{1}$ ∴ θ = OKA = 31°43′ nearly.

P. G. Tait fecit

2:2:[illegible]

P. G. Tait

$20^3 \quad 21^3$

$44100 \quad 9261$

$(210^3+21)^3$

$1^3 + 2^3 + 3^3 + 4^3$

$1 + 8 + 27 + 64 + 125$

$1 \quad 9 \quad 36 \quad 100 \quad 225$

$1 - 3 - 6 - 10 - 15$

$2 \quad 3 \quad 4 \quad 5$

$\therefore \frac{u_1 - u}{h} = \frac{1}{2\sqrt{x+a}} - \frac{h}{8(x+a)^{\frac{3}{2}}} + \ldots$

$\therefore \frac{du}{dx} = \frac{1}{2\sqrt{x+a}}$

$x\sqrt{x+a} = u$

$x^3 + ax^2 = u^2$

$3x^2 + 2ax\,dx = 2u\,du$

$\frac{du}{dx} = \frac{3x^2+2ax}{2x\sqrt{x+a}}$

$u = x\sqrt{x+a}$

$u_1 = (x+h)\sqrt{x+h+a}$

$\therefore \frac{u_1 - u}{h} = \frac{x}{2\sqrt{x+a}} + \sqrt{x+a} + \ldots$

$\therefore \frac{du}{dx} = \frac{3x+2a}{2\sqrt{x+a}}$

£3.

2d.

$\cos\theta : -\cos(\alpha + \theta) :: m : n$

$\therefore \cos\theta = -m(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$

$\therefore n = -m(\cos\alpha - \sin\alpha\tan\theta)$

$\therefore \tan\theta = \dfrac{n + m\cos\alpha}{m\sin\alpha} = n\,\mathrm{cosec}\,\alpha + \cot\alpha$

$343 \,|\, 1029 \qquad 2p - \frac{a^2}{2}\tan\theta$

$3462 \,|\, 69282p - \frac{a^2}{8}\tan\theta$

$346410) \ldots (\cdot 08072$

$OK = \frac{a}{2}, \; KN = \frac{\sqrt{3}\,a}{2}, \; ON\sqrt{3} = b$

$PK = y$

$\frac{a^2}{8}\cos\theta\,\sqrt{3}\,b$ a minimum

$\frac{\sqrt{3}}{2}a^2\,d\theta\,\sin\theta\cos\theta =$

$\sqrt{3}\,a^2\cos\theta + \frac{a^2}{2}p - \frac{a^2}{2}\tan\theta$ minimum

$\sqrt{3}\,\sin\theta \ldots - \frac{a^2}{2}\,\frac{1}{\cos^2\theta} = 0$

$\sqrt{3}\sin^3\theta = 1,$

$\sin^2\theta = \frac{1}{3^{\frac{1}{2}}} = \frac{\sqrt{3}}{3} = \frac{1\cdot 7320508}{3}$

$= \cdot 5773502\ldots$

$(a+b+c)(a^2+b^2+c^2)$

$b^2\ (3b+a+c)$

$-b^2\ (a-b+c)$

$a^2\ (3a+b+c)$

$-b^2$

$b^2\times 4b + a^2\times 4a + c^2\times 4c$

$4(a^3+b^3+c^3)$

$(c+d)^2+(b-a)^2$

$2(c+d)^2+2\ \ ^2(b^2-a^2)^2$

$(a+b)^2+(a-b)^2=$

$2(a-b)^2$

$2(c-d)^2+2\ (a+b)^2$

$a^2-x^2+b(a+x)$

$a^3-ax^2-a^2x+x^3$

$a^2-x^2,\ a^2-2ax+x^2$

$x^2+2ax+x^2$

$(a-1)x^3-(b-p)x^2+(c+q)x-(d+r)$

$a^3+ab^2+ac^2+ba^2+b^3$

$8\ (x^2+y^2)$

n^2-n

$\frac{n^4-1}{12}$

$(n^2+1)(n^2-1)$

$(n)(n^2+1)(n^2-1)$

$c^2+2cd+d^2$

$c^2-2cd+d^2$

$2c^2+2d^2$

Find the volume of a paraboloid [illegible]

Find the Chord of curvature [illegible] forms of a Parabola by Limits

(1.) Explanation of [illegible] being necessary [illegible] continuously [illegible]

[illegible] that the difference between the quantities compared by Limits [illegible] the [illegible] quantities [illegible] as its limit [illegible]

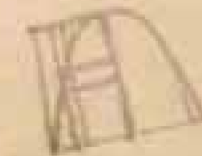
(2) & (3) Prove that the circumscribed [illegible] figure of the [illegible]

(4) [illegible] the same number of [illegible] alternately [illegible] the inscribed [illegible] in that ratio.

[illegible] Direct problem of Differentiation

[illegible] ABC : ABD [illegible] when [illegible] PP. [illegible]

[illegible]



[illegible]

A quantity of any fraction increased by 1 [illegible] by its reciprocal increased by 1 is the [illegible] itself.

$1+2x-x^5-17x^6+15x^7 \div$

by $1+4x+7x^2+10x^3+13x^4+[illegible]$

Whether is greater x^3+y^3 or x^2y+xy^2

x^2-xy+y^2

$x+y)\,x^3+y^3\,(x^2-xy+y^2$

x^3+x^2y

$-x^2y+y^3$

$-x^2y-xy^2$

xy^2+y^3

Cs is 85, As 55, Bs 65

B − C = 5 B + C = 60

Bs £34, Cs £26, As £5 6

5 minutes + $\frac{5}{11}$ minute to 11 o'clock

[illegible] the difference is constant [illegible]

[illegible] proprietor [illegible]

or each must lose £ 66.

[illegible]

$\frac{p}{q} + \frac{q}{s}$

$\frac{m+1}{\frac{1}{m}+1} = m$

No. 1.

$\frac{\frac{p}{s}+1}{1+\frac{s}{p}} = \frac{p}{s}$

We are conscious that $x^2+y^2 > 2xy$. ∴ $x^2-2xy+y^2 > xy$.

$\frac{x^3+y^3}{x+y}$ & $\frac{x^2y+xy^2}{x+y}$ = respectively

x^2-xy+y^2 & xy but the former of these is the greater ∴ $x^3+y^3 > x^2y+xy^2$.

Q. E. D.

The product of any fraction increased by 1 & by its reciprocal increased by 1 is the f[illegible] itself.

$$1+2x-x^5-17x^6+15x^7 \div$$

by $1+4x+7x^2+10x^3+13x^4+$ [illegible]

$$\frac{x-y}{x^2-xy}$$

A. Summit of the principal Axis

NAB = 90°

x^2-xy

$x-y$

AB. Axis

CD. Plane of Section

$x=y \therefore x^2=xy$

$x^2-xy=xy-y^2$

$\therefore x^2-y^2=xy-y^2$

$x=y$

$x+y=y$

$\therefore x=0$

$\frac{dV}{dx} = 4\pi by \therefore V = 4\pi b \int y \, dx \quad V = 4\pi b \frac{\pi a c}{2}$

$\therefore$ solidity $= 2\pi^2 b a c$

By Guldinus Theorem

solidity $= 2b\pi \times \pi a c = 2b\pi^2 a c$

$\begin{array}{r} 2.0 \\ 3 \cdots \\ \hline 6 \cdots \end{array}$ 3.1415926

$x^{n-1} \pm a x^{n-2} + \frac{n-3}{2} a x^{n-3}$

$x - m$

$x^n \mp a x^{n-1} + \frac{n-3}{2} a x^{n-2}$

$\pm m x^{n-1} + am \, x^{n-2}$

~~$x^n - (a+m) x^{n-1} + \left(\frac{n-3}{2} + a\right) a x^{n-2}$~~

$x^n - (a+m) x^{n-1} + \left(\frac{n-3}{2} + m\right) a x^{n-2}$

$f(x) = Q(x-a) + R$

$x^{n-1} - (p-a) x^{n-2} + (q - ap) x^{n-3}$

$x+a) \; x^3 - a^2 x + a^2 \ldots +$

$x^3 - 5x^2 + 14 - x \; (x - 2$

$x^3 - 7x - 2x^2$

$-2x^2 + 6x + 14$

$-2x^2 + 6x + 14$

(0)

$-x^2 + 2ax + a^2 \; (-x + a$

$-x^2 + ax$

$+ ax + a^2$

$+ ax - a^2$

$+ 2a^2$

$x - a) \; x^n - p x^{n-1} + q x^{n-2} \; (x^{n-1}$

$x^n - a x^{n-1}$

$(-p + a) x^{n-1} + q x^{n-2}$

$(-p + a) x^{n-1} + a(p - a) x^{n-2}$

$+ q - p + a) x^{n-2}$

$x^{n-1} - (p - a) x^{n-2} + (q - p + a) x^{n-3} +$

$(r - q + p - a) x^{n-3} + \&c$

[illegible] the moment [illegible] [illegible]
we have the [illegible] [illegible]
On [illegible] of figure [illegible]
[illegible] $OS = 1000$ [illegible]

OG = [illegible]
[illegible]

[illegible]
[illegible]
[illegible]

[illegible]) [illegible] ([illegible]

[illegible]

∴ the centre of gravity [illegible] miles
from the [illegible]
This might evidently have been [illegible]
assuming $SG = x$ & forming an equation
of the moments on the opposite sides.

II.

As the distance [illegible]
[illegible]
[illegible]

[illegible]

[illegible]

Centre of gravity of the [illegible]

[illegible] as $1:2$ [illegible]
[illegible]

Apply Guldin's Theorem to find the
mass of Saturn's Ring supposing its
section to be an [illegible] [illegible]
ellipse whose axes are 200 & 3000
miles & its mean radius 75000
miles.

[illegible] 75000

Friday Morning

Therefore volumes are as $1^3 : 2^3 = 1 : 8$ ∴ the
densities $\frac{\text{as weights}}{\text{volumes}}$ [illegible] $\frac{2}{1} : \frac{1}{8}$
$= 16 : 1$.
which was to be found

III.

$AB = b$
$AB = a$
$PL = y$
$BN = x$ $Bm = c$

then the [illegible] by $PL = \frac{b}{a}(\ldots)$
[illegible]

$\frac{1}{1+x^2}$ $(1+x^2)^{-1} = 1 - x^2 \ldots$

$(1+x^2)\ 1\ (1 - x^2 + x^4 - x^6 + \ldots$

$-x^2 - x^4$

$+x^4$

$x - \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 5}$

$1 - \frac{1}{3} + \frac{1}{5}$

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$

$= \frac{\pi^2}{6}$

$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} = \frac{\pi^2}{24}$

$\frac{1}{4} - \frac{1}{16} + \frac{1}{36} - \frac{1}{64} = \frac{\frac{\pi^2}{2}}{24}$

$= \frac{\pi(\pi - 1)}{24}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{36} + \frac{1}{72}$

$\frac{1}{3} - \frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \frac{1}{27}$

3 1 4 1 4 2 1 2 7

24) 1 0 0 2 7 2 3 9 1 7

1 8 8 7 0 0 4 7

2 8 2 6

60400

56564

1207

$\frac{7}{1} + \frac{2}{9} + \frac{2}{25} + \frac{2}{49}$ $\frac{\sqrt{2}}{1} + \frac{1}{9\sqrt{2}}$

$\frac{\pi}{2} = \sqrt{2} + \frac{\sqrt{2}}{9}$

$\frac{1}{162}$

$1 + \frac{1}{9}$

1.414

3.14

$\frac{20}{9}\sqrt{2}$ $\frac{19\sqrt{2}}{9}$

[illegible]

9) 1 6 0 8 0 2

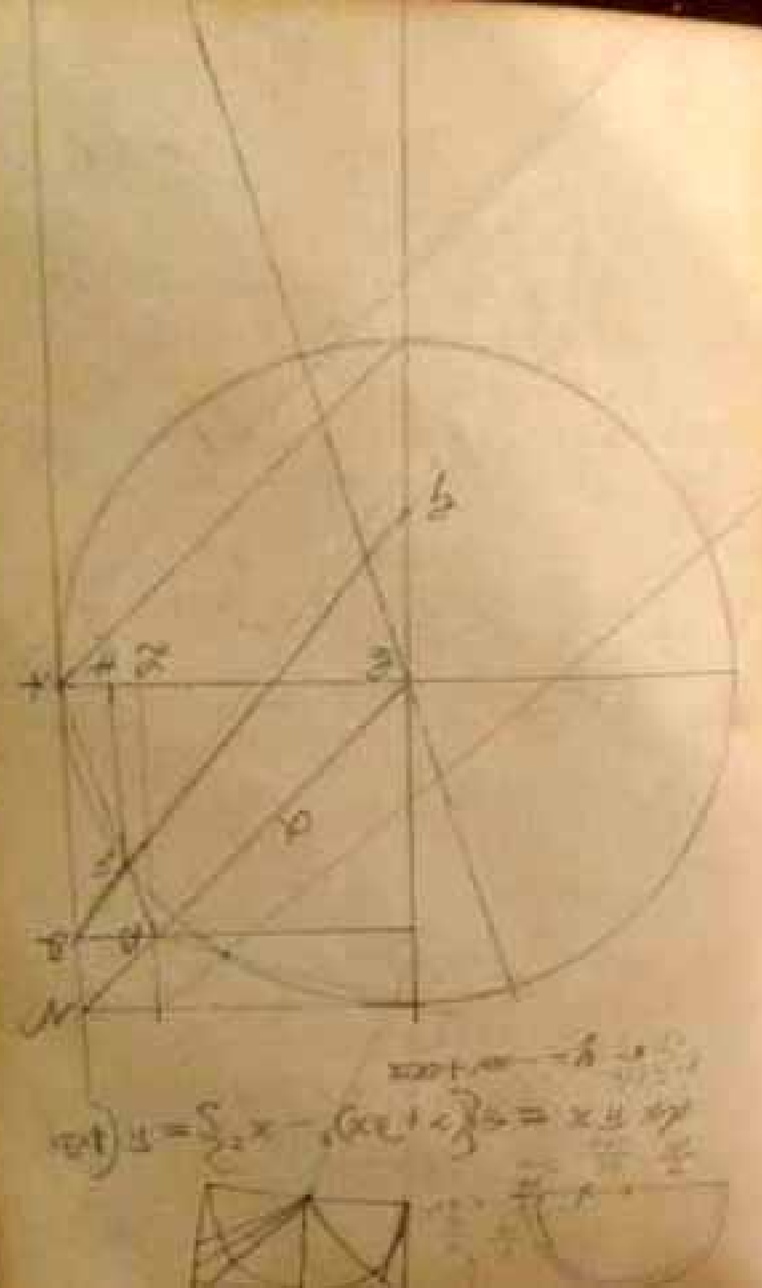
$1 - \frac{1}{3} + \frac{1}{5} + \ldots$

$1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{25}$

1 8 2 2 9

6 1 7 3 8 4

6 1 7 2 4 4 1 · 1



[illegible]

&c.

... the log of 10 is 1.

∴ The log of 1 is 0.)

Prop. The log of a quotient is the diff of the logs.

$\log\left(\frac{7}{8}\times 8\right) = \log\frac{7}{8} + \log 8.$

But $\frac{7}{8}\times 8 = 7 \therefore \log 7 = \log\frac{7}{8} + \log 8$

$\therefore \log 7 - \log 8 = \log\frac{7}{8}.$

Prop. Given the log to find the number.

What no has $\frac{1}{2}$ for its log?

$\frac{1}{2} + \frac{1}{2} = 1.$

$\therefore \log + \log = \log 10.$

$\therefore \frac{1}{2} = \log \sqrt{10}.$

Cont'd [illegible]

[illegible]

A [illegible] position of the [illegible]

$\tan\varphi \propto$ as weight of curve
if when $\varphi = \frac{\pi}{2}$, weight [illegible]
weight of curve $= a \tan\varphi$

$W = a \tan\varphi$
$dw = a \sec^2\varphi\, d\varphi$
$\frac{dw}{d\varphi} = a \sec^2\varphi$

$l = a\varphi$
$W : l :: OAP : OAR$

$\tan\varphi = \frac{x}{\sqrt{a^2 - x^2}}$
$W : l :: \frac{xa^2}{\sqrt{a^2 - x^2}} : \frac{a^2}{2} \sin^{-1}\frac{x}{a}$

$\frac{W}{l} \tan^{-1}\frac{x}{a} = \frac{ax}{\sqrt{a^2 - x^2}}$

[illegible]

For number of vibrations in a given time is as $\frac{1}{l}\sqrt{w}$

where l is the length & w the weight which gives tension to the string.

let φ = weight [illegible] point p

$\int y\, d(\sin^{-1}\frac{y}{a}) = \frac{ax}{\sqrt{a^2 - x^2}}$

$\frac{y}{\sqrt{a^2 - x^2}} = \frac{\sqrt{a^2 - x^2} + x\frac{x}{\sqrt{a^2 - x^2}}}{a^2 - x^2}$

$y = \frac{a^2 \sqrt{a^2 - x^2}}{(a^2 - x^2)^{\frac{3}{2}}}$

$\frac{y}{\sqrt{a^2 - x^2}} = \frac{a\left(\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2 - x^2}}\right)}{a^2 - x^2}$

$y = \frac{a(a^2)}{a^2 - x^2} = \frac{a^3}{a^2 - x^2}$

$t^2 = x + a$

[illegible]

$u = \sqrt{x+a} = \sqrt{x} + \frac{a}{2\sqrt{x}} - \frac{a^2}{8\sqrt{x^3}} + \frac{a^3}{16\sqrt{x^5}}$

$u_1 = \sqrt{(x+h)+a} = \sqrt{x+h} + \frac{a}{2\sqrt{x+h}} -$ [illegible]

$\frac{u_1 - u}{h} = \sqrt{x+h} - \sqrt{x} + \frac{a}{2\sqrt{x+h}} - \frac{a}{2\sqrt{x}} -$ [illegible]

$\frac{u_1 - u}{h} =$ [illegible] $+ \frac{a}{2\sqrt{h}} - \frac{a^2}{8\sqrt{h^3}}$ [illegible]

[illegible]

[illegible]

[illegible] Force [illegible] to the [illegible]

Fig. of Rational Mechanics [illegible]

[illegible] computation

[illegible] unit of force

[illegible] acting upon [illegible] force in a given direction is proportional to the cos of the angle made with the [illegible] direction of that force.

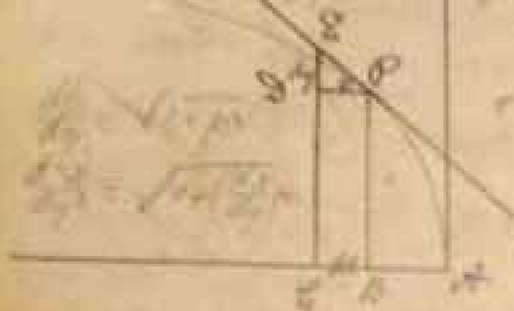
B. When 3 forces [illegible] are [illegible] each is proportional to the sine of the angle between the other two.

α α α α α α α α α α α α

α α α α α α α ∞ ∞

α ϵ γ δ γ ϵ α

ζ η θ ι κ κ κ



The strain or weight at Q is as $\frac{dy \cdot dx}{dy}$ ... as $\sqrt{1+\left(\frac{dx}{dy}\right)^2}$ & the integral of this must be proportional to $\frac{dx}{dy}$ ∴

$$m\frac{dx}{dy} = \int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$m \frac{d^2x}{dy} = \sqrt{dy^2 + dx^2}$$

$$\tan\varphi \cdot x = m \cot\varphi$$

$\tan^2\varphi \cdot \frac{x}{2}$ = m being the Parameter

weight is $= a\frac{dx}{dy}$

but the strength (& the weight) of PG is $= a\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

[redacted]

$$\frac{dx}{dy} = \int \sqrt{dy^2 + dx^2}$$

or the tangent of the angle of deflection is as the length of the curve

Fagnano's Property of the Ellipse.

Let $CA = 1$, $CB = b$,

[redacted] $\sqrt{1-b^2} = c$,

$CS = p$, $DS = t$,

$ACS = CHK = \varphi$, and arc $AD = z'$.

$CH^2 \cdot \cos^2\varphi = b^2(1 - CH^2 \sin^2\varphi)$ ∴ $CH = \frac{b}{\sqrt{1-c^2\sin^2\varphi}}$

$= \frac{b}{p}$ ∴ $p = \sqrt{1-c^2\sin^2\varphi}$ ∴ $z' + t =$

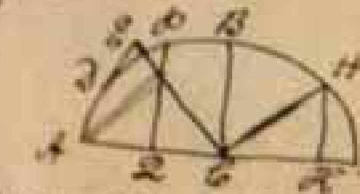
$\int d\varphi \sqrt{1-c^2\sin^2\varphi}$.

take $CQ = CA \cdot \sin QCA$. Let $z = AQ$

$z = \int d\varphi \sqrt{1-c^2\sin^2\varphi}$ ∴ $z = z' + t$ or

they can only differ by a constant.

$$z - z' = \frac{c^2 \sin\varphi \cdot \cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}.$$



$$s = \frac{a^2}{h}\left(\tan\frac{1}{2}v + \frac{1}{3}\tan^3\frac{1}{2}v\right).$$

$$\frac{dx}{dy} = \int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dx$$

$$\frac{d^2x}{dy^2} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dx$$

$$\frac{d^2x^2}{dy^2} = dx^2 + \frac{dx^4}{\ldots}$$

cosines [illegible] changing the products of cosines into the cosines of simple arcs in addition after all reductions

$$xy = 2(\cos 3\varphi + \cos 4\varphi + \cos 6\varphi + \ldots$$

or $$xy = -2(\cos 15\varphi + \cos 13\varphi + \cos 11\varphi + \ldots + \cos\varphi),$$

or at last $xy = -1$.

From these two equations we find

$$x = \tfrac{1}{4} + \tfrac{1}{4}\sqrt{17}, \quad y = \tfrac{1}{4} - \tfrac{1}{4}\sqrt{17}.$$

Now if we divide anew the second x & y into two parts viz.

$$x = s + t \qquad y = u + z$$

$$s = \cos 3\varphi + \cos 5\varphi. \qquad u = \cos\varphi + \cos 13\varphi.$$

$$t = \cos 7\varphi + \cos 11\varphi. \qquad z = \cos 9\varphi + \cos 15\varphi.$$

we obtain in like manner

$$st = -\tfrac{1}{4}, \qquad uz = -\tfrac{1}{4}.$$

From these 4 equations we may determine s, t, u, z.

Lastly knowing $(\cos\varphi + \cos 13\varphi) = u$ &

$\cos\varphi . \cos 13\varphi = \tfrac{1}{2}(\cos 14\varphi + \cos 12\varphi) = -\tfrac{1}{2}(\cos 3\varphi + \cos 5\varphi) = -\tfrac{1}{2}s$, we may obtain the value of $\cos\varphi$ & thence the side of the polygon which is

$$2\sin\varphi = 2\sqrt{1-\cos^2\varphi}.$$

$\left(\frac{\varphi}{2}\right)^{4} x\sqrt{2ax^3+x^4}$

$\partial\left(\omega^{\varphi}\sqrt{2ax^3+x^4}\right) = \sqrt{2ax^3+x^4}\,\partial x$

$\left(\omega^{\varphi-1}\varphi\,\partial\omega + \omega^{\varphi}\lambda\omega\,\partial\varphi\right)\sqrt{2ax^3+x^4} +$

$\omega^{\varphi}\,\dfrac{6ax^2\partial x + 4x^3\partial x}{2\sqrt{2ax^3+x^4}}$.

now let $u = f(x)$ & $\varphi = F(x)$

$f(x)^{F(x)-1}\,F(x)\,\partial f(x) + f(x)^{F(x)}\,\lambda f(x)\,\partial F(x)$

$+ f(x)^{F(x)}\,\dfrac{6ax^2\partial x + 4x^3\partial x}{2(2ax^3+x^4)} = \partial x$

let $u = \omega^{\varphi}$

$\dfrac{du}{dx} + u\,\dfrac{3ax^2+2x^3}{2ax^3+x^4} = 1$

$du + u(\,)\,\partial x = \partial x$

[illegible] $+ x^{5}\partial x + 3au x^2\partial x + 2ux^3\partial x =$

$2ax^3 dx + x^5\partial x$

$\dfrac{2ax^4}{4} + \dfrac{x^5}{5}$

$xz = \sqrt{a}.\cot.\phi,$ 4.
$yz = \sqrt{a}.\cot.\tau,$ 5.
6.

and by substituting in the other equations

$\sqrt{a}\,(\tan.\phi + \cot.\phi) = b,$
$\sqrt{a}\,(\tan.\psi + \cot.\psi) = c,$
$\sqrt{a}\,(\tan.\tau + \cot.\tau) = d.$

Now by the formula quoted for last question

$\tan.A\,(\text{any arc}) + \cot.A = 2\,\text{cosec}.2A = \frac{2}{\sin 2A}$

therefore $\sin.2\phi = \frac{2\sqrt{a}}{b},$ 7.

$\sin.2\psi = \frac{2\sqrt{a}}{c},$ 8.

$\sin.2\tau = \frac{2\sqrt{a}}{d}.$ 9.

From these equations we find $\phi,\ \psi$ &c.

By multiplying together the corresponding sides of 1, 2 & 3,

$u^2xyz = (\sqrt{a})^3\tan.\phi.\tan.\psi.\tan.\tau;$

dividing this by equation (A)

$u^2 = \sqrt{a}\,(\tan.\phi.\tan.\psi.\tan.\tau)$

or $u = \sqrt[4]{a}.\sqrt{\tan.\phi.\tan.\psi.\tan.\tau}.$

By a similar process we find

$x = \sqrt[4]{a}\sqrt{\cot.\phi.\cot.\psi.\tan.\tau}$

$y = \sqrt[4]{a}\sqrt{\cot.\phi.\tan.\psi.\cot.\tau}$

$z = \sqrt[4]{a}\sqrt{\tan.\phi.\cot.\psi.\cot.\tau}$

3. To inscribe in a circle a regular polygon of 17 sides.

Let the arc $\frac{\pi}{17} = \varphi$, then,

$(\cos\varphi + \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \cos 9\varphi + \cos.11\varphi + \cos 13\varphi + \cos 15\varphi) = \frac{1}{2}.$

For putting P for the first member & multiplying all the terms by $2\cos\varphi$ & transforming by the formula $(2\cos a.\cos b = \cos(a-b) + \cos(a+b))$ (from 7 page 1) we have $2P\cos\varphi =$

$(1 + 2\cos 2\varphi + 2\cos.4\varphi + 2\cos.6\varphi + 2\cos.8\varphi + 2\cos.10\varphi + 2\cos 12\varphi + 2\cos 14\varphi + \cos 16\varphi)$

But since $17\varphi = \pi \therefore \cos 2\varphi = \cos(\pi - 15\varphi) = -\cos.15\varphi.\ \cos.4\varphi = -\cos.13\varphi$ & so on to $\cos.16\varphi = -\cos\varphi \therefore 2P\cos\varphi =$

$1 - 2\cos.15\varphi - 2\cos.13\varphi$ * * * $-2\cos.3\varphi - \cos\varphi$, or $2P.\cos\varphi = 1 + \cos\varphi - 2P$ or $2P(1+\cos\varphi) = 1 + \cos\varphi \therefore P = \frac{1}{2}.$

This being proved we now divide the terms composing P into two parts.

$x = \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \cos.11\varphi.$

$y = \cos\varphi + \cos.9\varphi + \cos.13\varphi + \cos 15\varphi.$

$\therefore x + y = \frac{1}{2}.$

Next multiply the 4 terms of x &c

$v^2 - 2v.\cos.\frac{\pi}{m}+1$,

$v^2 - 2v.\cos.\frac{3\pi}{m}+1$, * * *

* * * * $v^2 - 2v.\cos.\frac{3\pi}{m}+1$,

$v^2 - 2v.\cos.\frac{\pi}{m}+1$.

Therefore when m is an even number v^m+1 is = the product of

$v^2 - 2v\cos.\frac{\pi}{m}+1$, * * *

$v^2 - 2v.\cos.\frac{3\pi}{m}+1$, * * *

$v^2 - 2v.\cos.\frac{5\pi}{m}+1$, $v^2 - 2v\cos.\frac{m-1}{m}\pi+1$.

But when m is odd the factors are

$v+1$, * * *

$v^2 - 2v.\cos.\frac{\pi}{m}+1$,

$v^2 - 2v.\cos.\frac{3\pi}{m}+1$, $v^2 - 2v.\cos.\frac{m-2}{m}\pi+1$.

We shall now give three examples of the application of the Calculus of Sines.

1. To solve the quadratic equation

$x^2 + px = q$, where p & q are both +.

Let $\tan.\frac{1}{2}v\sqrt{q}$ be the positive root where v is an angle to be determined then $-\cot.\frac{1}{2}v\sqrt{q}$ will be the negative root for $(\tan.\frac{1}{2}v\sqrt{q} \times -\cot.\frac{1}{2}v\sqrt{q}) = -q$. To determine v we have the equation $\sqrt{q}(\cot\frac{1}{2}v - \tan.\frac{1}{2}v) = p$ but by the last two formulae on the 3rd page of this article

$\cot.\frac{1}{2}v - \tan.\frac{1}{2}v = 2\cot.v = \frac{2}{\tan.v}$;

$\therefore \tan v = \frac{2\sqrt{q}}{p}$, hence to solve equation

$x = +\tan.\frac{1}{2}v\sqrt{q}$ & $-\frac{\sqrt{q}}{\tan.\frac{1}{2}v}$.

---

when the equation is

$x^2 - px = q$,

$x = +\frac{\sqrt{q}}{\tan.\frac{1}{2}v}$; & $x = -\sqrt{q}\tan.\frac{1}{2}v$.

---

when the equation is

$x^2 - px = -q$ the roots are both positive and may be expressed by $\tan.\frac{1}{2}v\sqrt{q}$ & $\cot.\frac{1}{2}v\sqrt{q}$. In this case by the formulae lately referred to

$(\cot.\frac{1}{2}v + \tan.\frac{1}{2}v) = 2\operatorname{cosec}.v = \frac{2}{\sin v} = \frac{p}{\sqrt{q}}$

$\therefore \sin.v = \frac{2\sqrt{q}}{p}$;

Then $x = +\sqrt{q}.\tan.\frac{1}{2}v$ & $x = +\frac{\sqrt{q}}{\tan.\frac{1}{2}v}$.

---

2. Find u, x, y, z from the equations

$uxyz = a$ (A)

$uz + xy = b$ (B)

$uy + xz = c$ (C)

$ux + yz = d$ (D)

Solution.

Let p, q, r be such angles that

$uz = \sqrt{a}\tan.p$ 1.

$uy = \sqrt{a}\tan.q$ 2.

$ux = \sqrt{a}\tan.r$ 3.

then from equation (A)

... n has the form $4m+1$ take the upper & when the form $4m+3$ the lower sign. In either case the series terminates with a multiple ...

$$\pm 2^{n-1}\sin^n a = \sin na - n\sin(n-2)a + \frac{n(n-1)}{1\cdot 2}\sin(n-4)a - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\sin(n-6)a \mp \&c$$

When $n = 4m+2$ take the upper & when ... the lower sign. Continue the series to ... $\cos 0a$ $(=1)$ & divide its numeral coefficient by 2.

$$\mp 2^{n-1}\sin^n a = \cos na - n[\cos(n-2)a] + \frac{n(n-1)}{1\cdot 2}\cdot\cos(n-4)a - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cos(n-6)a + \&c$$

$$\cos na \pm \sqrt{-1}\cdot\sin na = (\cos a \pm \sqrt{-1}\sin a)^n$$

Which is true whether n is whole or fractional positive or negative

$$\cos na = \frac{(\cos a + \sqrt{-1}\sin a)^n + (\cos a - \sqrt{-1}\sin a)^n}{2}$$

$$\sin na = \frac{(\cos a + \sqrt{-1}\sin a)^n - (\cos a - \sqrt{-1}\sin a)^n}{2\sqrt{-1}}$$

From this by the binomial theorem we derive

$$\cos na = x^n - \frac{n(n-1)}{1\cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)(n-3)}{\ldots}x^{n-4}y^4 - \&c$$

$$\sin na = \frac{n}{1}x^{n-1}y - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n-5}y^5 - \&c$$

$$\cos a = \frac{e^{a\sqrt{-1}} + e^{-a\sqrt{-1}}}{2}, \quad \sin a = \frac{e^{a\sqrt{-1}} - e^{-a\sqrt{-1}}}{2\sqrt{-1}}$$

where $e = 2.7182818$ &c

$$a = \tan a - \frac{1}{3}\tan^3 a + \frac{1}{5}\tan^5 a - \frac{1}{7}\tan^7 a + \&c$$

Theorem.

Let (n) be any whole number, & let (n^2+1) be resolved into any two factors p & q one of which may be unity. The arc whose tangent $= \frac{1}{n}$ is $=$ the sum of the arcs whose tangents are $\frac{1}{n+p}$ & $\frac{1}{n+q}$.

Let $n = 0$

$$\tan^{-1}\frac{1}{0} = \tan^{-1}\frac{1}{1} + \tan^{-1}\frac{1}{1}.$$

or the arc whose tangent is infinity (that is $90°$) is $=$ twice the arc whose tangent is 1 $(45°)$.

$\pi = 3.14159265358979323846264338327950288 \ldots$

$$\frac{1}{a} = \frac{1}{\tan a} + \frac{1}{2}\tan\frac{1}{2}a + \frac{1}{4}\tan\frac{1}{4}a + \frac{1}{8}\tan\frac{1}{8}a + \&c$$

$\cos a = x$ & $\sin a = y$.

$2\cos na = (2x)^n - n(2x)^{n-2} + \frac{n(n-3)}{1\cdot 2}(2x)^{n-4} - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}(2x)^{n-6} + \&c.$

$\sin na = y\left[(2x)^{n-1} - (n-2)(2x)^{n-3} + \frac{(n-3)}{1}\cdot\frac{(n-4)}{2}(2x)^{n-5} - \frac{(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3}(2x)^{n-7} \&c\right]$

when n is odd, = 4m+1 upper, n=4m+3 the lower sign

$\cos na = \pm\left[nx - \frac{n(n^2-1)}{2\cdot 3}x^3 + \frac{n(n^2-1)(n^2-9)}{2\cdot 3\cdot 4\cdot 5}x^5 - \&c\right]$

when n is even, = 4m+2 upper sign, = 4m lower do.

$\cos na = \pm\left[1 - \frac{n^2}{2}x^2 + \frac{n^2(n^2-4)}{2\cdot 3\cdot 4}x^4 - \frac{n^2(n^2-4)(n^2-16)}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \&c\right].$

When n is = 4m+1 the upper, & when = 4m+3 lower sign.

$\sin na = \pm y\left\{1 - \frac{n^2-1}{2}x^2 + \frac{(n^2-1)(n^2-9)}{2\cdot 3\cdot 4}x^4 - \frac{(n^2-1)(n^2-9)(n^2-25)}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \&c\right\}.$

But when n = 4m+2 the upper, & when = 4m lower &

$\sin na = \pm\left\{nx - \frac{n(n^2-4)}{2\cdot 3}x^3 + \frac{n(n^2-4)(n^2-16)}{2\cdot 3\cdot 4\cdot 5}x^5 - \&c\right\}.$

when n is odd.

$\cos na = x\left\{1 - \frac{n^2-1}{2}y^2 + \frac{(n^2-1)(n^2-9)}{2\cdot 3\cdot 4}y^4 - \frac{(n^2-1)(n^2-9)(n^2-25)}{2\cdot 3\cdot 4\cdot 5\cdot 6}y^6 - \&c\right\}$

$\sin na = ny - \frac{n(n^2-1)}{2\cdot 3}y^3 + \frac{n(n^2-1)(n^2-9)}{2\cdot 3\cdot 4\cdot 5}y^5 - \&c$

When n is even.

$\cos na = 1 - \frac{n^2}{2}y^2 + \frac{n^2(n^2-4)}{2\cdot 3\cdot 4}y^4 - \frac{n^2}{2}\cdot\frac{(n^2-4)(n^2-16)}{3\cdot 4\cdot 5\cdot 6}y^6 + \&c$

$\sin na = x\left\{ny - \frac{n(n^2-4)}{2\cdot 3}y^3 + \frac{n(n^2-4)}{2\cdot 3}\cdot\frac{n^2-16}{4\cdot 5}y^5 - \&c\right\}$

If α, β, γ, δ &c represent the coefficients of the second, third, fourth, fifth &c terms of a binomial raised to the nth power then

$\tan na = \frac{\alpha t - \gamma t^3 + \epsilon t^5 - \&c}{1 - \beta t^2 + \delta t^4 - \&c}.$

$2^{n-1}\cos^n a = \cos na + n(\cos(n-2)a) + \frac{n(n-1)}{1\cdot 2}\cos(n-4)a + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cos(n-6)a + \&c$

this is to be continued till we arrive at a negative arc, & if n be an even no. the half of the coefficient of cos 0a (which = 1) is to be taken

$$\cos b + \cos a = 2\cos\tfrac{1}{2}(a+b).\cos\tfrac{1}{2}(a-b).$$
$$\cos b - \cos a = 2\sin\tfrac{1}{2}(a+b).\sin\tfrac{1}{2}(a-b).$$
$$\sin a + \sin b = 2\sin\tfrac{1}{2}(a+b).\cos\tfrac{1}{2}(a-b).$$
$$\sin a - \sin b = 2\cos\tfrac{1}{2}(a+b).\sin\tfrac{1}{2}(a-b).$$

$$2\cos a\cos na = \cos(n-1)a + \cos(n+1)a.$$
$$2\sin a.\sin na = \cos(n-1)a - \cos(n+1)a.$$
$$2\cos a.\sin na = \sin(n+1)a + \sin(n-1)a$$
$$2\sin a.\cos na = \sin(n+1)a - \sin(n-1)a$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a.\tan b}.$$

$$\cot(a \pm b) = -\frac{1 \mp \cot a.\cot b}{\cot a \pm \cot b}$$

$$\tan a.\tan b = \frac{\cos(a-b) - \cos(a+b)}{\cos(a-b) + \cos(a+b)}.$$

$$\tan a.\cot b = \frac{\sin(a+b) + \sin(a-b)}{\sin(a+b) - \sin(a-b)}.$$

$$\cot a.\tan b = \frac{\sin(a+b) - \sin(a-b)}{\sin(a+b) + \sin(a-b)}.$$

$$\cot a.\cot b = \frac{\cos(a-b) + \cos(a+b)}{\cos(a-b) - \cos(a+b)}$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a.\cos b}.$$

$$\cot a \pm \cot b = \frac{\sin(a \pm b)}{\sin a.\sin b}$$

$$\tan a + \cot b = \frac{\cos(a-b)}{\cos a.\sin b}.$$

$$\cot a - \tan b = \frac{\cos(a+b)}{\sin a\cos b}.$$

$$\sin^2 a - \sin^2 b = \sin(a+b)\sin(a-b).$$
$$\cos^2 a - \sin^2 b = \cos(a+b)\cos(a-b).$$

$$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\tan\tfrac{1}{2}(a+b)}{\tan\tfrac{1}{2}(a-b)}.$$

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \tan\tfrac{1}{2}(a+b).$$

$$\frac{\sin a + \sin b}{\cos a - \cos b} = \cot\tfrac{1}{2}(a-b).$$

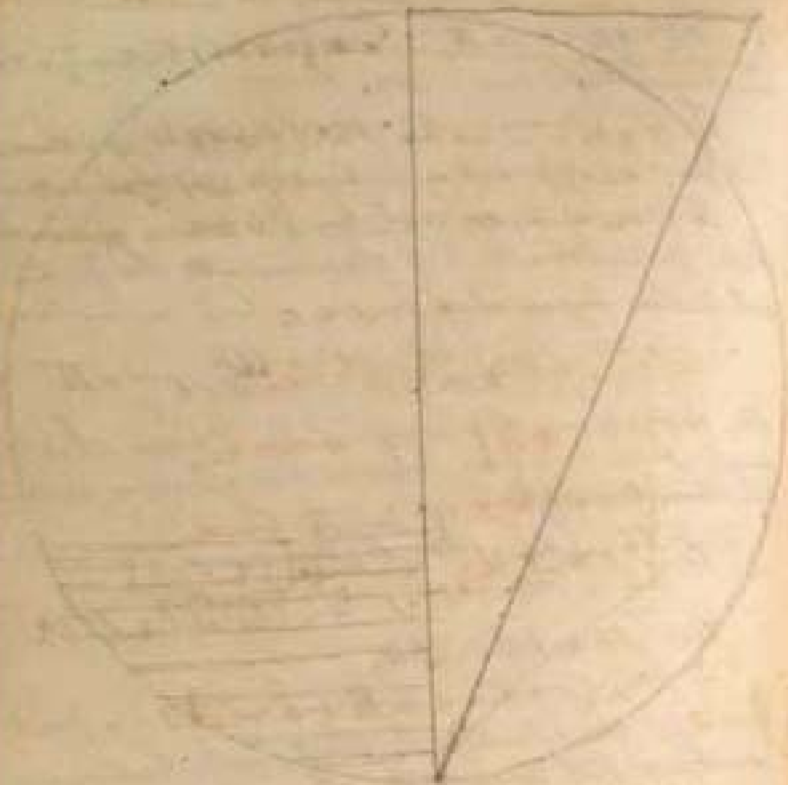
$$\frac{\sin a - \sin b}{\cos a + \cos b} = \tan\tfrac{1}{2}(a-b).$$

$$\frac{\sin a - \sin b}{\cos a - \cos b} = -\cot\tfrac{1}{2}(a+b).$$

$$\frac{\cos b + \cos a}{\cos b - \cos a} = \frac{\cot\tfrac{1}{2}(a-b)}{\tan\tfrac{1}{2}(a+b)}.$$

$$\sin b.\sin(a-c) = \sin c.\sin(a-b) + \sin a.\sin(b-c)$$
$$\cos b.\sin(a-c) = \cos c.\sin(a-b) + \cos a.\sin(b-c)$$

$$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$$
$$\sin 2a = 2\sin a.\cos a.$$
$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a} = \frac{2}{\cot a - \tan a}.$$
$$\cot 2a = \tfrac{1}{2}(\cot a - \tan a).$$
$$\cos\tfrac{1}{2}a = \sqrt{\frac{1+\cos a}{2}} = \tfrac{1}{2}\left\{\sqrt{1+\sin a} + \sqrt{1-\sin a}\right\}$$
$$\sin\tfrac{1}{2}a = \sqrt{\frac{1-\cos a}{2}} = \tfrac{1}{2}\left\{\sqrt{1+\sin a} - \sqrt{1-\sin a}\right\}$$
$$\tan\tfrac{1}{2}a = \frac{\sin a}{1+\cos a} = \operatorname{cosec} a - \cot a.$$
$$\cot\tfrac{1}{2}a = \frac{\sin a}{1-\cos a} = \operatorname{cosec} a + \cot a.$$



Formulae derived from the Arithmetic of Sines with a few examples of their application.

$\cos.(\pi \pm a) = -\cos.a. \qquad \sin.(\pi \pm a) = \mp \sin.a.$

$\cos.(2\pi \pm a) = \cos.a. \qquad \sin.(2\pi \pm a) = \pm \sin.a.$

$\cos.(3\pi \pm a) = -\cos.a. \qquad \sin.(3\pi \pm a) = \mp \sin.a.$

&c. &c.

and in general (n being a whole number)

$$\cos.\left[(2n+1)\pi \pm a\right] = -\cos.a.$$

$$\cos.\left[2n\pi \pm a\right] = +\cos.a.$$

$$\sin.\left[(2n+1)\pi \pm a\right] = \mp \sin.a.$$

$$\sin.\left[2n\pi \pm a\right] = \pm \sin.a.$$

$$c = a.\cos B + b.\cos A.$$

$$\cos.(A \pm B) = \cos.A.\cos B \mp \sin A.\sin B.$$

$$\sin.(A \pm B) = \sin A.\cos B \pm \cos A.\sin B.$$

$$\cos.a.\cos.b = \tfrac{1}{2}\left[\cos(a-b)\right] + \tfrac{1}{2}\left[\cos(a+b)\right].$$

$$\sin.a.\sin.b = \tfrac{1}{2}\left[\cos(a-b)\right] - \tfrac{1}{2}\left[\cos(A+B)\right].$$

$$\sin.a.\cos.b = \tfrac{1}{2}\left[\sin(a+b)\right] + \tfrac{1}{2}\left[\sin(a-b)\right].$$

$$\cos.a.\sin.b = \tfrac{1}{2}\left[\sin(a+b)\right] - \tfrac{1}{2}\left[\sin(a-b)\right].$$

Making $(a+b) = A$ & $(a-b) = B$, substituting, & changing A & B into a & b, we obtain, [illegible] in the opposite side of C from A, the other conditions remaining the same, OB is negative. If also we the direction in this prop. remains unchanged. But if we be [illegible] $(CP_1)(CP_2)\ldots$ &c $= OA^n + OB^n$. [illegible] remarks that when lines as OA are in the original direction, since the coefficient of direction in that case is unity it is immaterial whether we write OA or (OA).

Ex. Let $n = 3$ & $OB = \frac{1}{2}$,

then, $(AB) = \frac{3}{2}$, $(CP_1) = (CP_2) = \frac{\sqrt{7}}{2}$

$\therefore (CA)(CP_1)(CP_2) = \frac{3}{2} \cdot \frac{\sqrt{7}}{2} \cdot \frac{\sqrt{7}}{2} = \frac{9}{8} = 1 + \frac{1}{8}$

$R^3 + \frac{1}{2}^3 = OA^3 + OB^3$.

Cor. 2. If C be in OA produced the reasoning & result will be the same as in the prop. only, since now CA & CB being of the same affection -1 is not a divisor of the second member of the Equation, &c

$(CP_1)(CP_2)\ldots$ &c $= (OB)^n - (OA)^n$.

---

11. If from A the extremity of the diameter (Fig. 5) the circumference be divided into n equal parts & the several extremities be joined, then

$(AP_1)(AP_2)\ldots(AP_{n-1}) = n\,CA^{n-1}$

As in former prop. $AP_1 = CP_1 - CA$, $AP_2 = CP_2 - CA$ [illegible]

$\therefore AP_1 \cdot AP_2 \ldots AP_{n-1} = \overline{CP_1 - CA} \cdot \overline{CP_2 - CA}$ [illegible]

$= R^{n-1} \left\{ S_{n-1} - S_{n-2} \ldots \pm S_1 \pm 1 \right\}$ where $S_1 S_2$ &c [illegible] the sums, sum of products two & two, &c of all the coefficients except unity there being no line drawn from A to the circumference in the direction CA. S_1, S_2 &c are coefficients of the Equation $\frac{x^n - 1}{x - 1}$ or of $x^{n-1} + x^{n-2}$ &c [illegible] with the signs changed for the product of all [illegible] changed for [illegible].

If $\frac{n-1}{2}$ be even $S_{n-1} = \pm 1$, $S_{n-2} = \ldots$ [illegible]

if $\frac{n-1}{2}$ be odd $S_{n-1} = \ldots$, $S_{n-2} = \pm 1$ [illegible]

$\therefore AP_1 \cdot AP_2$ &c $= R^{n-1} \times \pm \{1 + 1 + 1 + 1 \ldots$ [illegible] $\} = nR^{n-1}$

according as $\frac{n-1}{2}$ is even or odd.

If $\frac{n-1}{2}$ be even, $AP_1 \cdot AP_2$ &c $= (AP_1)(AP_2)$ &c the several pairs of coefficients giving unity for their products.

If $\frac{n-1}{2}$ be odd then the several pairs give only for their product unity but there remains the factor [illegible] which has for its coefficient -1.

$\therefore$ in either case $(AP_1)(AP_2) \ldots (AP_{n-1}) = nR^{n-1}$

---

12. If by this method we undertake to prove that the angles at the base of an Isosceles triangle are equal to each other we have $(AC) = (BC)$ (Fig. 6).

But $AC = (AC) \cdot 1^{\frac{a}{\pi}} = (AC) \cdot [a + \sqrt{-b}]$

$CB = AB - (AC) \cdot 1^{-\frac{a'}{\pi}} = (AC) \cdot (a' + \sqrt{-b'})$.

But $AC + CB = AB$.

$\therefore (AC) \cdot (a + a' + \sqrt{-b} + \sqrt{-b'}) = AB =$ a positive quantity consequently the imaginary parts destroy one another or $\sqrt{-b} = -\sqrt{-b'}$ or $b = -b'$. Therefore the angles A & B have their lines of equal length but of different affections. The angles themselves being together less than π are geometrically equal to each other.

Cor. Much in the same way we might prove that in every triangle the greater side has the greater angle opposite to it & conversely that the greater angle has the greater side opposite to it.

May 27th 1847.

P. G. Tait

$\sin \overline{A+B} = \sin A . \cos B + \cos A . \sin B$

$\cos \overline{A+B} = \cos A . \cos B - \sin A . \sin B$.

Let arc AB_1 (Fig 7) $= \mathbf{A}$, $AB_2 = AB_1$ each $= \mathbf{B}$.

then by Prop. 3, $CB_1 = r . 1^{\frac{A}{2\pi}}$

$CD_1 = r . 1^{\frac{B}{2\pi}}$, $CD_2 = r . 1^{\frac{A+B}{2\pi}}$

$\therefore CD_2 = r . 1^{\frac{A}{2\pi}} . 1^{\frac{B}{2\pi}}$;

But by Prop 7, $1^{\frac{A}{2\pi}} = \cos A + \sqrt{-1} \sin A$,

$1^{\frac{B}{2\pi}} = \cos B + \sqrt{-1} \sin B$,

$\therefore 1^{\frac{A+B}{2\pi}} = \cos A \times \cos B - \sin A \times \sin B + \sqrt{-1}$ (&c.)

$(\sin A . \cos B + \cos A . \sin B)$,

but $1^{\frac{A+B}{2\pi}} = \cos \overline{A+B} + \sqrt{-1} \sin \overline{A+B}$,

Equating then the real & unreal parts of these we have

$\cos A . \cos B - \sin A . \sin B = \cos \overline{A+B}$

$\sin A . \cos B + \cos A . \sin B = \sin \overline{A+B}$

Definition

It should be observed that in the following proposition a line expressed by letters simply as AB must be considered both as to length & direction while when in brackets thus (AB) its length alone is referred to. Thus $(AB) 1^{\theta} = AB$.

9. In any right angled triangle the sum of the squares of the sides is = square of hypotenuse.

Let OA (Fig 8) $= r$, then $CA_1 = r . 1^{\frac{\theta}{2\pi}}$ &c.

$CA_1 \times CA_{n-1} = r^2 \times 1^{\frac{\theta}{2\pi}} \times \frac{1}{1^{\frac{\theta}{2\pi}}} = r^2$;

$\therefore$ ...

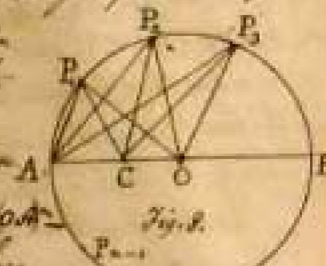
Also $CA_1 = (CD_1) + \sqrt{-1} (D_1A_1)$ ...

$= (CD_1) - \sqrt{-1} (D_1A_1)$ ...

CA_{n-1} ... $= (CD_1)^2 + (D_1A_1)^2$ ...

10. Cotes' Properties of the Circle.

If the circumference be divided into n equal parts and join OP_1, OP_2, OP_3 &c (Fig 8) and also join P_1, P_2, P_3 with C any point in the diameter. Then



$CP_1 = OP_1 - OC$, $CP_2 = OP_2 - OC$ &c

$\therefore CP_1 ; CP_2 ; CP_3 \ldots CP_n = \Sigma_n . OA^n - \Sigma_{n-1} (OA)^{n-1} \ldots \pm OC^n$ where

Σ_n is the product of all the coefficients of direction for OP_1, (the principle of ...) OP_2 &c, Σ_{n-1}, the sum of these coefficients taken $n-1$ together &c. But these coefficients are also the roots of the equation $x^n - 1 = 0$. Now the product of the roots of the equation with their signs changed is -1 & Σ_n is $=$ the product with their signs unchanged. Therefore if n be even $\Sigma_n = -1$ but if odd $= +1$, and in either case Σ_{n-1}, Σ_{n-2} &c each $= 0$. Hence $CP_1 . CP_2 . CP_3 \ldots CP_n = \pm OA^n \mp OC^n$ the upper signs to be used when n is even, the lower when odd.

Here CP_1, CP_2 &c consider the lines both as to length and direction, we must $\therefore$ divide the first or multiply the second by the product of all the coefficients of direction. If n be even the several pairs as CP_1, CP_{n-1} are evidently of the form $(CP_1) 1^{\frac{\theta}{2\pi}}$ and $(CP_{n-1}) \frac{1}{1^{\frac{\theta}{2\pi}}}$ $\therefore CP_1 \times CP_{n-1} = (CP_1) \times (CP_{n-1})$ and this is true for every pair except $CA = (CA)$ and $CB = (CB) . -1$ $\therefore (CP_1) . (CP_2) \ldots CP_n = (-OA^n + OC^n) . -1 = OA^n - OC^n$

But if n be odd the several pairs remain as before only no P falling on B, -1 is not a coefficient of direction

$\therefore (CP_1) . (CP_2)$ &c $= OA^n - OC^n$ as before.

6\. In the foregoing Props & examples it has been taken for granted that we know not only the several roots of unity but also their proper order; that is the order in which as coefficients they express the radii drawn to the extremities of the arcs ϑ, 2ϑ, 3ϑ &c with the original radius. But when we determine the roots of $x^n - 1 = 0$ we obtain them in no fixed order. To determine this order we must observe that two roots are always of the form $a \pm \sqrt{-b}$ comparing which with (Fig 6) a is evidently the part symbolical of the cosine & $\sqrt{-b}$ that of the sine because it is affected by $\sqrt{-1}$ and is $\therefore$ perpendicular to original radius. Thus $\therefore$ in $a \pm \sqrt{-b}$, $+$ refers to radii in the upper semicircle & $-$ to those in the under; and that the two radii whose symbols differ only in the sign of $\sqrt{-b}$ are at equal angles to the original radius on opposite sides of it. $\therefore$ the root in which a is greatest is nearest to the original radius. Thus the roots of $x^6 - 1$ arranged properly are

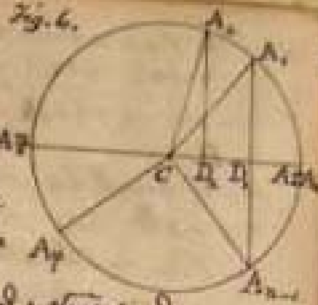
$$1, \frac{1+\sqrt{-3}}{2}, \frac{-1+\sqrt{-3}}{2}, -1, \frac{-1-\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}.$$

symbolizing the radii respectively drawn to the arcs of

$$0 \text{ or } 360°, 60°, 120°, 180°, 240°, 300°.$$

For if $+1$ be first -1 having no cosine part must be the middle. Next $\frac{1+\sqrt{-3}}{2}$ & $\frac{-1+\sqrt{-3}}{2}$ must be in the upper half of the circle and $\frac{1+\sqrt{-3}}{2}$ must come first because its cosine is in 6A. And so with the rest.

7\. It appears from Props 4, 5 that the radius drawn to the arc $p\vartheta$ is $= 1 \angle p\vartheta$ and this again by $a \pm \sqrt{-b}$ where a is relative to the cosine & $\sqrt{-b}$ the sine of $p\vartheta$. Since (Fig 6) let $A_0A_1 = \vartheta$, $A_0A_p = p\vartheta$ whichever $p\vartheta$, then



$$CA_1 = CD + \sqrt{-1}\cdot DA_1 = \cos\vartheta + \sqrt{-1}\cdot\sin\vartheta,$$

$$CA_p = \cos p\vartheta + \sqrt{-1}\sin p\vartheta,$$

But by prop 2,

$$CA_p = \overline{CA_1}^{\,p} = (\cos\vartheta + \sqrt{-1}\sin\vartheta)^p$$

$$\therefore (\cos\vartheta + \sqrt{-1}\sin\vartheta)^p = \cos p\vartheta + \sqrt{-1}\sin p\vartheta,$$ which is

Demoivre's Theorem.

Cor. If $p\vartheta = 2\pi$, $\cos p\vartheta + \sqrt{-1}\cdot\sin p\vartheta = 1$.

Hence $(\cos\vartheta + \sqrt{-1}\cdot\sin\vartheta)$, $(\cos 2\vartheta + \sqrt{-1}\cdot\sin 2\vartheta)$, &c represent the several p^{th} roots of unity. If we arrange the angles, instead of $\vartheta, 2\vartheta, 3\vartheta$ &c, in pairs thus ϑ & $\overline{p-1}\,\vartheta$, 2ϑ & $\overline{p-2}\,\vartheta$, the several expressions for x — the several p^{th} roots of unity or the simple factors of $x^p - 1 = 0$ taken in pairs corresponding with the above will be

$$(x - \cos\vartheta - \sqrt{-1}\cdot\sin\vartheta) \;\&\; (x - \cos\overline{p-1}\,\vartheta - \sqrt{-1}\cdot\sin\overline{p-1}\,\vartheta)$$

which last is $= (x - \cos\overline{p\vartheta - \vartheta} - \sqrt{-1}\cdot\sin\overline{p\vartheta-\vartheta})$

$$= (x - \cos\overline{2\pi - \vartheta} - \sqrt{-1}\cdot\sin\overline{2\pi-\vartheta}) = (x - \cos\vartheta + \sqrt{-1}\sin\vartheta)$$

In the same way the next pair must be

$$(x - \cos 2\vartheta + \sqrt{-1}\sin 2\vartheta) \;\&\; (x - \cos 2\vartheta - \sqrt{-1}\sin 2\vartheta)$$

Multiplying these together for the quadratic factors of $x^p - 1$ we obtain when p is even

$$x^p - 1 = (x^2 - 1)(x^2 - 2x\cos\vartheta + 1)(x^2 - 2x\cos 2\vartheta + 1) \text{ \&c to } \tfrac{p}{2} \text{ terms}$$

But when p is odd $x^p - 1 = (x - 1)(x^2 - 2x\cos\vartheta + 1)$ &c to $\frac{p+1}{2}$ terms

where ϑ it may be observed is $= \frac{2\pi}{p}$.

3. If we now assume the several lines CA, CA_1, CA_2 &c all equal or = radii of a circle the case will not be altered. Let n be a divisor of 2π or let $\vartheta = \frac{2\pi}{n}$. Then the Radius $a^n = a^{\frac{2\pi}{\vartheta}}$ is the same in length & position as CA ∴ $a^n = 1^n = 1^{2\pi}$, (as known from ordinary Algebraical principles that the several nth roots of unity may be expressed by the lines CA, CA_1, CA_2 &c.

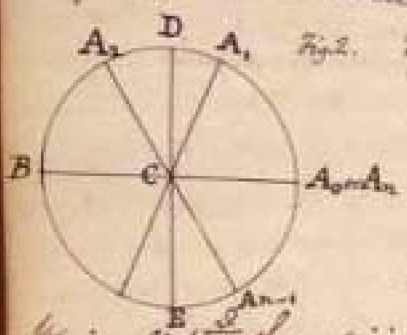


Fig. 2.

It therefore follows that we may take the successive radii of a circle at equal angles for the several roots of unity & conversely. If R be the [illegible] of radius that radius will just be first at $\vartheta = R \times 1^{2\pi}$.

We ∴ call $1^{\frac{\vartheta}{2\pi}}$ the coefficient of direction because it refers only to the direction, never to the length of a line. Thus $a \times \frac{1+\sqrt{-3}}{2}$ is a line = a simply.

4. Let us next suppose $n = 2$, AB will be a diameter & if $CA = 1$, $CB = -1$. But $a^2 = 1$ ∴ $a = \pm 1$. But the radii being a, a^2, a must evidently be $= -1$ [illegible].

Next let $n = 4$, CA, CD, CB, CE are the 4 roots of the equation $a^4 - 1 = 0$. But these roots are ± 1 & $\pm\sqrt{-1}$. Hence CA & CB are symbolized by $+1$ & -1 respectively ∴ CD & CE must be symbolized by $+\sqrt{-1}$ & $-\sqrt{-1}$ respectively, it being however quite optional which direction from C we account positive & negative either in the horizontal or perpendicular lines.

5. It appears from the foregoing proofs that if a [illegible] both its length & direction [illegible] symbolized by $a . 1^{\frac{\vartheta}{2\pi}}$ we have

$a . 1^{\frac{\vartheta}{2\pi}}$ ∴ represents the actual transference of the point in space by moving from A to B. But it is also (Fig. 3) clear that its actual transference in space though not its distance travelled would be the same did it move from A to B_1 & then from $B_1 C$. Thus ∴ (AC × its coefficient of direction) = (AB × its coefficient of direction) + (BC × its coefficient of direction) or the sum of any two lines making an angle with each other is = the diagonal of the parallelogram completed. Even in this it [illegible] from it is only the general [illegible] proposition [illegible] $AB + BC = AC$.

Fig. 3.

1. As examples to elucidate this let ABC (Fig. 4) be an isosceles right angled triangle described on the radius AD. If we note AD symbolized by $a . 1^0 = a$ & $= a \times \frac{1+\sqrt{-1}}{\sqrt{2}}$, but $AC = \frac{a}{\sqrt{2}}$, CB being perpendicular & [illegible] is $= \frac{a}{\sqrt{2}} \times \sqrt{-1}$ (Fig. 4) ∴

$AC + CB = a\left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a$

$= a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB$.

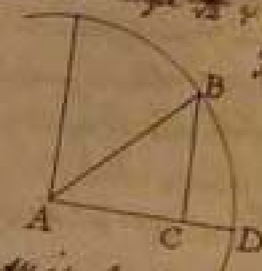


Fig. 4.

2. Let $BAD = 60°$, $BCD = 90°$, then AB in length & direction is $a . 1^{\frac{1}{6}} = a . 1^{\frac{1}{6}} = a . \frac{1+\sqrt{-3}}{2}$, $AC = \frac{a}{2}$, CB in length $= \frac{a\sqrt{3}}{2}$ ∴ in length & direction jointly $= a . \frac{\sqrt{3} . \sqrt{-1}}{2} = a \frac{\sqrt{-3}}{2}$.

∴ $AC + CB = \frac{a}{2} + a . \frac{\sqrt{-3}}{2} = a . \frac{1+\sqrt{-3}}{2} = AB$.

3. Let the triangle (Fig. 5) be equilateral & let AB be the original position. Let $AB = a$, $AC = a . 1^{\frac{1}{6}}$, $CB = a . 1^{-\frac{1}{6}}$ ∴ $AC + CB$

$= a . \left[1^{\frac{1}{6}} + 1^{-\frac{1}{6}}\right] = a . \left[1^{\frac{1}{6}} + \frac{1}{1^{\frac{1}{6}}}\right] = a . \left[\frac{1^{\frac{1}{3}} + 1}{1^{\frac{1}{6}}}\right] =$

$= a . \left[\frac{-1+\sqrt{-3}}{2} + 1\right] \times \frac{2}{1+\sqrt{-3}} = a . \left[\frac{1+\sqrt{-3}}{2} \times \frac{2}{1+\sqrt{-3}}\right] = a$

$= AB$.

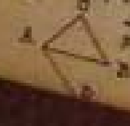


Fig. 5.

[illegible] line when angle of inclination [illegible] is a^n & [illegible] ... 3

3. If we now assume the several lines CA, CA_1, CA_2 &c. all equal or the radii of a circle the case will not be altered. Let n be a divisor of $2r\pi$ or let $\theta = \frac{2r\pi}{n}$. Then the Radius $a^n = a^{\frac{2r\pi}{\theta}}$ is the same in length & position as CA $\therefore a^n = 1^n = 1^{\frac{\theta}{2\pi}}$ [illegible] from ordinary Algebraic principles that the several n^{th} roots of unity may be expressed by the lines a, a^2, a^3 &c.

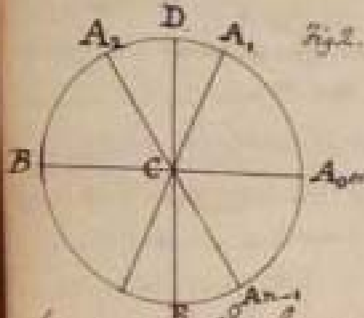


Fig. 2.

It therefore follows that we may take the [illegible] radii of a circle at equal angles for the several roots of unity & conversely. If R be the numerical length of radius that radius inclined [illegible] the first at θ is $= R \times 1^{\frac{\theta}{2\pi}}$.

We $\therefore$ call $1^{\frac{\theta}{2\pi}}$ the coefficient of direction because it applies only to the direction, never to the length of a line. Thus $a \times \frac{1+\sqrt{-3}}{2}$ is a line $= a$ simply.

4. Let us next suppose $n = 2$, AB will be a diameter & of CA $= 1$, CB $= -1$. But $a^2 = 1 \therefore a = \pm 1$. But the radii being a, a^2, a must evidently be $= -1$ [illegible].

Next let $n = 4$, CA, CD, CB, CE are the 4 roots of the equation $a^4 - 1 = 0$. But these roots are ± 1 & $\pm\sqrt{-1}$. Hence CA & CB are symbolized by $+1$ & -1 & $\therefore$ CD & CE must be symbolized by $+\sqrt{-1}$ & $-\sqrt{-1}$ respectively, it being however quite optional which direction from C we account positive or negative either in the horizontal or perpendicular lines.

5. It appears from the foregoing props. that if [illegible] length & direction [illegible] be $a \cdot 1^{\frac{\theta}{2\pi}}$ we know both its length & direction

$a \cdot 1^{\frac{\theta}{2\pi}} \therefore$ represents the actual transference of the point in space by moving from A to B. But it is also

Fig. 3.

clear that its actual transference in space though not its distance travelled must be the same did it move from A to B_1 & then from B_1 to C. Hence (AC $\times$ its coefficient of direction) $=$ (AB $\times$ its coefficient of direction) $+$ (BC $\times$ its coefficient of direction). $=$ the sum of any two lines making an angle with each other is $=$ the diagonal of the parallelogram completed. Even in this elementary [illegible] it is only the general assertion of a proposition [illegible] we admit when we say $AB_1 + B_1C$ [illegible] AC or that $AC + CB_1 = AB_1$.

1. As examples to illustrate this let ABC (Fig. 4) be an isosceles right angled triangle described on the radius AB. If we call AB [illegible] each side [illegible] will be $= $ length $\frac{a}{\sqrt{2}}$ & AB is symbolized by $a \times 1^{\frac{1}{8}} = a \times 1^{\frac{1}{8}}$

$= a \times \frac{1+\sqrt{-1}}{\sqrt{2}}$, but AC $= \frac{a}{\sqrt{2}}$

CB being perpendicular to original position is $= \frac{a}{\sqrt{2}} \times \sqrt{-1}$ (Prop. 4) $\therefore$

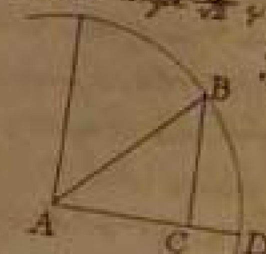


Fig. 4.

$AC + CB = a\left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a$

$= a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB$.

2. Let $\angle$ BAC $= 60°$, BCA $= 90°$, then AB its length & direction is $a \cdot 1^{\frac{1}{6}} = a \cdot 1^{\frac{1}{6}} = a \cdot \frac{1+\sqrt{-3}}{2}$, AC $= \frac{a}{2}$, CB in length $= \frac{a\sqrt{3}}{2} \therefore$ in length & direction jointly $= a \cdot \frac{\sqrt{3}\cdot\sqrt{-1}}{2} = a \cdot \frac{\sqrt{-3}}{2}$.

$\therefore AC + CB = \frac{a}{2} + a \cdot \frac{\sqrt{-3}}{2} = a \cdot \frac{1+\sqrt{-3}}{2} = AB$.

3. Let the triangle (Fig. 5) be equilateral & let AB be the original position. Let AB $= a$, AC $= a \cdot 1^{\frac{1}{6}}$, CB $= a \cdot 1^{-\frac{1}{6}}$ $\therefore$ AC + CB

$= a\left[1^{\frac{1}{6}} + 1^{-\frac{1}{6}}\right] = a\left[1^{\frac{1}{6}} + \frac{1}{1^{\frac{1}{6}}}\right] = a\left[\frac{1^{\frac{1}{3}}+1}{1^{\frac{1}{6}}}\right] =$

$= a\left[\frac{-1+\sqrt{-3}}{2}+1\right] \times \frac{2}{1+\sqrt{-3}} = a \cdot \left[\frac{1+\sqrt{-3}}{2} \times \frac{2}{1+\sqrt{-3}}\right] = a =$ AB.

Fig. 5.

[illegible] 1846.

$a + b = 900$

$a + c = 200$ requires [illegible]

$b + c = 300$

$200 - a = 300 - b$ $\quad 1 = \epsilon^{2x} - \epsilon^{y}$

$a = b - 100$

$a = 900 - b$ $\quad \epsilon^{y} = \epsilon^{2x} - 1$

$b = £500$ $\quad 1 = \epsilon^{2x-y}$ [illegible]

$a = £400$ $\quad \epsilon^{2x+y} - \epsilon^{2y} =$

$c = -£200$ $\quad = \epsilon^{2x} - [illegible]$

Therefore C's worth is £200 [illegible]

[illegible] $\quad \epsilon^{2x} - \epsilon^{y} [illegible]$

= [illegible] worth [illegible]

[illegible] the [illegible]

[illegible]

[illegible] It may mean (may [illegible])

[illegible]

$a = 1 - 1 + 1 - 1 + 1 - 1 + [illegible]$

[illegible]

$a = 1 - 2 + 3 - 4 + 5 = \frac{1}{(1-1)^2}$

$1 + 1 + 1$ to infinity

$-1 - 1 - 1$ to infinity

$\left(\frac{1+2}{2+1}\right) 1 \quad \frac{2+4}{2+2} = \left(\frac{5}{2}\right) 2$

$\frac{10(3-1)}{12(2-1)} = \frac{10}{12} = \frac{5}{6}(3)$

$(243)\,4 \quad (59049)\,6$

$(4^{25})\,4 \quad (0)\log \omega = \frac{\log(\epsilon^{2x}-1)}{2x}$

$(2a^6)\,[illegible] \quad 2a^4$

$\omega = \sqrt[2x]{\epsilon^{2x}-1}$

$4a^4$

$y = \int_x \sqrt{\epsilon^{2x}-1}$

Let $\omega^{2x} + 1 = \epsilon^{2x}$

$y = \int \omega^{x}\,dx = \frac{\omega^{x}}{\log \omega}$

$= \frac{\sqrt{\epsilon^{2x}-1}}{\frac{\log(\epsilon^{2x}-1)}{2x}}$

$\frac{\sqrt{\epsilon^{2x}-1}\,\log(\epsilon^{2x}-1)}{2x}$ $\quad \log(\epsilon^{2x}-1)$

3
9
27
81
243
729
2187
6561
19683
59049
177147
531441
1594323
4782969
14348907
43046721

$\left(3486784401\right) 5$

∴ A is on the curve. In the same way we may prove B & C to be on the curve, & it is evident from Prop II that if we take a less constant than c the foci will be without, but if greater within, the curve.

Which was to be done

Proposition IV.

Problem

The powers of the Foci & the constant being given, to cut an indefinite straight line in three points so that if the three be taken as foci and a trifocal curve be described with the said constant, the two extreme points may be on the curve.

Let $2c$ = const = c, m n & o the powers of the Foci in their order take any point B and cut off $AB = \frac{(m+n-o)c}{mn+n^2+no}$ and $BC = \frac{(n+o-m)c}{mn+n^2+no}$ then $AC = \frac{2c}{a+b+c}$, A, B & C are the Foci.

For $nAB = \frac{(m+n-o)c}{m+n+o}$ & $oAC = \frac{(2o)c}{a+b+c}$ ∴ $nAB + oAC = \frac{(m+n+o)c}{m+n+o} = c$, or A is on the curve. In the same way C may be proved to be on the curve.

Which was to be done.

Proposition V.

Problem.

To determine the angle formed at a Focus on a trifocal curve.

Let A (power m) be on the curve, let B & C (powers respectively n & o) be the other Foci.

With the same constant quantity & same powers of B & C describe an Oval of Descartes SAD it will of course pass through A. Take a point D indefinitely near A, Join DB, DC, AB, AC. Then because D is very near A, BAC = BDC. Draw a ~~tangent~~ AD & the perpendicular to it (or Normal) AT. Make PDO = KAT ∴ TAV = ODL. Join OB, OC then nBO + OCO + mAO = nBO + oCO cut off BP = BO & CL = CO, then ∴ PBO, OCL are very small, PO, OL are respectively perpendicular to DB, DC, also $nPD + oLD = mAO$. Draw TO ∥ AD & KT, TV perpendicular to AB, AC, then AT = DO, AK = PD & AV = DL ∴ nAK + oAV = mAO, and the same can be proved of the other side of A ∴ AN = AO & AON is a very small isosceles triangle coinciding with the curve.

∴ having drawn the Oval of Descartes & its ~~tangent~~, draw AT perpendic. & NTO ∥ AD. Draw TK, TV perpendic to AB, AC & make mAO = nAK + oAV. Make NA = AO, NAO is the angle required.

Which was to be done.

On Trifocal Curves.

Definitions.

1. Trifocal curves are those which are described by the motion of a point in a plane so that m, n & o times respectively its distances from A, B & C in that plane shall together be = constant.
2. A, B & C are called the Foci.
3. As before m, n & o are called the powers of the Foci.
4. When two of the foci are equidistant from the 3rd and have equal powers the curve is called Isosceles.

Proposition I.
Problem.

To describe a trifocal curve, the Foci, their powers & the constant quantity being given.

Let A, B, C be the Foci 1, 2 & 3 the powers & EF the constant quantity it is required to describe the curve.

Erect focal cylinders as before take a thread EF wrap it round the Foci & a moveable cylinder D so that there may be one ply between A & D, two between B & D & 3 between C & D. Move D round keeping the thread tight its extremity will describe the curve required.

For at any point D,

AD + 2BD + 3CD = thread = constant.

Proposition II.
Theorem

A Focus is within, on or without the curve according as the sum of its distances from the other Foci multiplied respectively by their powers is <, = or > the constant quantity.

1st Let oAB + nBC < const, B is within the curve.

Bisect ABC by BO then oAO + nCO + mBO = EF, & BO = $\frac{oAO + nCO - oAB - nCB}{m}$ = $\frac{o(AO-AB)+n(CO-CB)}{m}$

2nd It is evident that C is on the curve where oAC + nBC = EF

3rd Let mBA + nCA > EF, Draw AP within BAC then AP = $\frac{mBA + nCA - mBP - nCP}{o}$ = $\frac{m(BA-BP)+n(CA-CP)}{o}$

2: E: D.

Proposition III.
Problem.

The powers of the 3 Foci (of which any two must for this purpose be greater than the third), and the constant quantity being given to construct a triangle such that if its angles be taken as Foci and a trifocal curve described with the same constant, all the Foci will be on the curve, if with a greater constant, within, and if with a less, without, the curve.

Let HL the constant be = C, the powers of the Foci m, n & o. Draw a line AB = $\frac{C(m+n-o)}{2mn}$ let A be the one and B the Focus whose power is m, and B that whose power is n. Complete the triangle so that AC = $\frac{C(m+o-n)}{2mo}$ & BC = $\frac{C(n+o-m)}{2no}$ (Euc. I. 23) ABC is the required triangle.

For nAB = $\frac{C(m+n-o)}{2m}$ & oAC = $\frac{C(m+o-n)}{2m}$ ∴

nAB + oAC = $\frac{C(m+n-o+m+o-n)}{2m}$ = $\frac{2mC}{2m}$ = C

Join BG and produce it to H, BH is a tangent, for suppose H to be the end of the rod take any point O, n CH + m GL m OH + n OA ∴ O is without the curve.

which was to be done

In Fig 8, GP is the Apsoid, GL a circle, & GL a meloid all these having A & B as foci.

Prop. VII.

Problem.

To draw a tangent to an Apsoid at any point in the same. the Foci & Ratio being given.

It is required to draw a tangent to the Apsoid at the point C. Join BG and draw a circle as in Prop 3. Join AD & produce it. Make AP : PD :: n : m. Join CP. Draw CH at right angles to it. Describe a circle through C, P, H & it was proved in (Prop. VII of Descartes Ovals) that if any point O be taken & DO, AO joined DO : AO :: m : n suppose O to be both in the circle & Apsoid, join BO then LO : BO :: n : m :: DO : AO ∴ LO = DO but LO < DO ∴ the circle is without the Apsoid & a tangent CT to the circle at C must also touch the Apsoid.

which was to be done.

Prop. VIII.

Problem

To draw a tangent to a meloid at any point C.

Case 1st. Let the curve be concave towards B. Describe the circle POCK (Prop. VII.), it will be wholly within the meloid. At C draw a tangent CT to the circle, it must also touch the meloid.

For let [illegible] be the tangent to the meloid, it must cut the circle (Euc. III. 16) and ∴ also the

Case 2nd. When the curve is convex towards B. Describe a circle O as before. Draw CT a tangent to this, it also touches the meloid.

For suppose a circle S to touch the curve internally it must touch O & also CT and any other line than CT would cut S & consequently the meloid.

which was to be done.

Scholium. See figs. 8 & 9. Let M be the point, Join AM, BM, cut off ME so that AM is to ME :: Power of B : Power of A. Join AE, bisect AME by MH Draw XY perpendicular to AE & make XMH = XHM, XM is the tangent required.

Prop. IX.

Theorem

If lines be drawn from the Foci to any point in the Meloid or Apsoid, the sines of the angles which they make with the perpendicular to the Tangent are to one another as the powers of the foci. Figs 8, 9.

For CY is the perpendicular to the Tangent and ∴ (Cor. Prop VIII. Ovals of Descartes), (AP : PD) :: m : n, Sin ACY : Sin BCY

Q. E. D.

Prop. II.

Same as Prop. II of [illegible] Ovals.

Prop. III.

Theorem.

If a circle be described with a Focus for a centre & [illegible] for a radius, the distance of any point in the curve from the other Focus is to its distance from the circle as Power of central Focus : Power of other.

Let [illegible], [illegible] be the circles, at any point G, BG : GO :: m : n & PG : GA :: m : n.

For m GA − n GB = constant diff., but m OA = const. diff., ∴ m GO = n GB & BG : GO :: m : n.

And n BP = constant diff., ∴ m GA = n GB and PG : GA :: m : n.

Q. E. D.

Cor. 1. If constant diff. = 0 the curve is a circle.

Cor. 2. If an oval of Descartes be described with the constant diff. for a constant sum, same foci & same powers as the Meloid and any line be drawn from A, & GO, OB, OB bisects the angle GBO = OBO.

For BG : GO :: m : n & BD : DO :: m : n ∴ BG : GO :: BD : DO ∴ BG : BD :: GO : DO & GBO = OBO (Euclid VI. 3).

Prop. IV.

Theorem.

When the less focus is on the curve an angle will be formed at it = that in the oval (Ovals of Descartes Prop. IV) [illegible] indefinitely small arc BB in the circle.

For take an [illegible] & ∠BT = TBP ∴ [illegible] = [illegible]

GBD = DBE (III. Cor. 2) [illegible] as it was for the oval.

Or it may be proved as it was at an infinite distance the

If the greater focus is at an infinite distance the appearance is like that in Fig. V. Q. E. D.

Prop. V.

Theorem.

If the distance between the greater Focus & the point where the axis cuts the Meloid be to the distance between that point & the less Focus in a greater proportion than the power of the greater Focus to that of the less the curve at that point is convex towards the greater focus but if the proportion is less it will be concave.

Let AB : BB :: p : q.

If AB : BB > m : n, CBE is convex to A but if < concave.

Take C & E near to B & GO = DE. Join GE, [illegible] the axis in O. Draw the circle CLE from B & [illegible] from A. Also [illegible] as in Prop. III.

Then [illegible]

[illegible] :: BD : BT :: BX : PT ∴ BB : BL :: BT : PT :: BB : BT − BB

BL : BP :: m : n.

As E & G are very near B, AB : BB :: AL : BL but [illegible]

but [illegible] :: AL : BL :: AD : BB :: p : q & [illegible]

[illegible] :: PGE : LGE :: q PE : p PE :: q : p :: PGO :

LGO :: q : p & PO : OL :: q : p and if p : q > m : n, OL : OP

> BL : BP and B is nearer to A than the straight line COE or CBE is convex towards A, but if p : q < m : n B is on the opposite side or it is concave towards A.

Q. E. D.

Prop. VI.

Problem.

To draw a Tangent to either of the curves from a focus without.

Take m a Power of Greater foc. & n a Power of less foc. & find the angle ABC (Prop. V of Descartes Ovals) [illegible] AB

[illegible] ALGEB [illegible]

Proposition IX.
Axiom.

If lines be drawn from the Foci to any point in the oval, the sines of the angles which they make with the perpendicular to the tangent are to one another as the powers of the Foci;

~~that~~ is, Sine DCE : Sine ACE :: Power of A : Power of B.

For, describe the circle CXT, as in Prop. VII, so that DT : TA :: DX : AX then LC the tangent to the circle is a tangent to the oval and CE the Radius is perpendicular to it, then by Prop. VIII, Corollary,

Sin. DCE : Sin. ACE (:: DT : TA) :: power of A : Power of B.

Q. E. D.

~~[illegible]~~ + $c - mr = n\{\sqrt{r^2 \ldots}$ [illegible]

$mr +$ $c - mr = n\sqrt{a^2 + r^2 \ldots \cos\theta}$

$m\sqrt{x^2+y^2} + n\sqrt{(c-x)^2 + y^2} = c$

[illegible]

$m^2x^2 + m^2y^2 + n^2 \ldots$ [illegible]

$(m^2+n^2)(x^2+y^2) + n^2(a^2 \ldots) - \ldots$

$- 2mn\sqrt{(x^2+y^2)(\ldots)}$ [illegible] $+ 2n^2(m^2 \ldots)$

$\therefore (m^2+n^2)^2(x^2+y^2)^2 \ldots$

$+ n^4(a^4 -$

$c^2 - 2cmr + m^2r^2 = a^2n^2 + r^2n^2 - 2an^2r\cos\theta$

[illegible] $= a^2$ ~~[illegible]~~ $- 2ar\cos\theta$

Meloid & Apioid.

Definitions.

If a point move in a plane so that (m times its distance from A in the plane) ~ (n times do. from B in the plane) = constant quantity it will describe a curve which if m (dist. from A) > n (distance from B) is called a Meloid and if vice versâ an Apioid.

The points A & B are called the Foci.

Axiom.

It is possible for a circle to touch any given curve internally.

Proposition I.
Problem.

With given foci, given powers & a given constant to describe a Meloid or an Apioid.

Take a straight & rigid rod AB moving round A as a centre and passing at B [illegible] infinitely small cylinder [illegible] at B and take a perfectly flexible & inextensible thread [illegible] constant; then [illegible] round B [illegible] [illegible] [illegible] [illegible] the extremity of C will describe the meloid or apioid required.

For take any point C, m BC + n CA = thread + mAB + mAC = m AB; take away m BC + n CB [illegible] = thread ~ m AB = constant quantity.

As $m > n$ A is the greater & B the less Focus.

Taking a point R nearer to B than D, join AR, & cut off AO = AD; join OD. Make RO = HR = AG, [illegible] in S, DS will be on B's side of OD. ∴ RS : RD :: [illegible] n : m, & mRS = nRD. But SR < RD < RO ∴ mRS < mRO ∴ mRO > nRD add to both nBR & [illegible] mAO ∴ mRA + nBR > nBD + mAD.

Q : E : D.

Proposition 6th.

Problem.

To draw a tangent to an oval from the lesser focus when without the curve.

Let as before m = Power of A & n that of B.

By Prop V. find the angle ADB. Upon AB [illegible] -cribe the segment of a circle containing an equal angle. Join B & G the point where it cuts the oval, BG is a tangent.

For, take in BG any point O not in the oval, then mAO + nOB > mAG + nGB ∴ O is without the oval.

2 : E : F.

Proposition VII.

Problem.

To draw a tangent to a given oval at any point in its circumference.

It is required to draw a tangent at G to the oval [illegible] from B the less focus [illegible] G A F, ratio m : n. From B [illegible] GA. Join [illegible] [illegible] a circle as in Prop III. Join AD produce it to D, then DG : GA :: m : n. Join AD and produce it. Bisect DGA by GO ∴ DO : OA ∴ (DG : GA) m : n. Draw GH perpendicular to CO, and describe the circle OGLK. OGK = [illegible] AGD = OGA + AGK [illegible] KD ∴ [illegible]

(GA : GD) :: DO : OA, and at any point L in the circle AL : DL :: [DO : OA (Cor. 6. F)] n : m. The circle is wholly without the oval & can only touch it in G & P where DG cuts the oval, for suppose them to coincide at any point L, join BL & produce to X, then,

$$\left.\begin{array}{l} AL : DL :: n : m \\ \text{But by Prop III. } AL : XL :: n : m \end{array}\right\} \therefore DL = XL, \text{ but}$$

(Cor III. 7) DL > XL ∴ the oval is within the circle.

Draw then, SGF a tangent to the circle at G and as it is completely without the circle except at G, it must also meet the oval at G & at no other point.

2 : E : F.

Scholium. Join AG, BG and produce till AG : GD :: n : m. Join AD. Bisect AGD by GO. Draw OE perpendicular to AO. Make OGS = SOG then it is evident from the proposition that SGF is the tangent required.

Proposition VIII.

Theorem.

If a line AB be cut in G & B so that AB : BG :: AE : GE and a circle be described with BE for a diameter, AO, AG, GO be drawn to O in the circumference and perpendiculars OH, DL be drawn to them from the centre. DH : DL :: AB : BG.

For AB : BG :: AG : GE ∴ AB : AG [illegible] :: 2DB : BG or AB : AD :: BD : BG, ∴ BD : GD [illegible] BG. Draw DT perpendicular to OB, and in OTP, DPL, OTP = DLP & OPT = DPL; then it [illegible] PDT = PDL, [illegible]

[illegible]

2 : E : D.

Proposition II.
Theorem.

The greater Focus is always within the oval, but the less is within or without it, according as the distance between the Foci & power of the greater Focus is <, =, or > the constant quantity.

1 The greater Focus is always within the Oval. For suppose it to be at A, (Fig. 2) then, m Ab + n Bb = constant = m AB + m Bb + n Bb, and m AB + n BB = constant + m AB + n Bb [illegible] = m AB + m Bb + n Bb ∴ n Bb = m Bb or m = n. But by Hypothesis m > n ∴ A cannot be without the curve, that is, it must be within it.

2 The less Focus B (Fig 3) is within the branch m AB < EF the constant. For it is evident that

$$Bb = \frac{EF - m\,AB}{m + n}.$$

It is on the curve when m AB = EF, this needs no demonstration.

It is without the curve when m AB > EF for

$$Ab = \frac{EF - n\,AB}{m - n}.$$

Proposition III.
Theorem.

If with one of the Foci for a centre, and the constant quantity divided by the power of the Focus for a radius, a circle be described, the distances of any point in the oval from the other focus, is to the distance from that point to the circumference of the circle measured on the radius through the point, as the power of the centre Focus is to that of the other.

The circle EHB is described with centre B and radius = constant EF / Power of B. At any point b, Ab : bB :

Power of B : Power of A, m = Power of A, n = Power of B; let m = Power of A [illegible] = n Bb + n bB [illegible] = constant quantity

BH [illegible] m = constant quantity

= m Bb + m Ab ∴ n bH = m Ab or Ab : bH :: m : n.

Q. E. D.

Cor. 1. When the powers of the Foci are equal, the curve is an Ellipse.

Cor. 2. When the less focus is at an infinite distance the curve is an Ellipse for the circle becomes a straight line.

Cor. 3. When the greater focus is at an infinite distance the curve is an hyperbola for the same reason.

Proposition IV.
Theorem.

When the less focus is on the curve an angle will be formed there = the vertical angle of an isosceles triangle of which the side is to the Perpendicular on the base as the power of the greater focus is to that of the less.

For let a circle be described as in Prop III. it is evident that it will pass through B. Take indefinitely small arcs bB = BD, join bA & DA, join EH.

EG : GB :: Power of B : Power of A, and BG = BD ∴

BD : BO :: Power of A : Power of B.

Q. E. D.

Proposition V.
Problem.

A point A without, and a point B on, a straight line BC, being given, to find a point D in the line so that m AD + n BD may be a minimum.

Take a line HA [illegible] HK perpendicular [illegible] as centre describe a circle with radius = m HK / n [illegible] so that m : n :: HA : HK.

Draw AD perpendicular to BC, & make [illegible] AD = HA [illegible] D is the point required.

For take any other point in BC, E for instance. Join AE [illegible] perpendicular [illegible] Join EF [illegible]

And in the triangles [illegible] angle in each ∴ [illegible] = m DE ∴ [illegible] m BE > m DE [illegible] and m EA [illegible] m DA [illegible] m BD + [illegible] m DA >

Prop. 2. Theorem.

Conical Pendula suspended from the same point revolve in the same time in the same horizontal plane, and if they revolve in different planes the squares of the times are as the perpendiculars from the fixed point upon the plane.

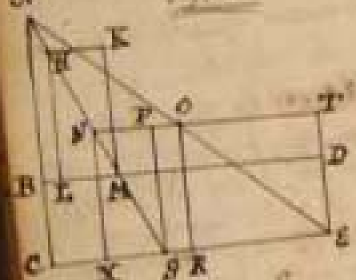
Let S be the force of gravity, then the centrifugal (tending towards) force is $= BX$, in the same way the force of gravity on C is represented by SX.

Then by last Prop. centrifugal force is as distance $\div$ square of time and centripetal force is $= it$ $\therefore \frac{XS}{CS} = \frac{1}{(\text{time of } S)^2}$ & $\frac{RE}{CE} = \frac{1}{(\text{time of } E)^2}$

But $AC : CS :: AX : XS$ & $AC : CE :: CK - AX : RE$

$\therefore CS : XS :: CE : RE \therefore$

Fig. 2.



$\therefore \frac{XS}{CS} = \frac{RE}{CE} \therefore$ time of S = time of E.

Q. E. D.

2nd. Let M & S be bodies revolving in different planes, [illegible] that centripetal force at [illegible] & as $LM = XS$ the centripetal force of M = that of S & $BM \div (M\text{'s time})^2 = CS \div (S\text{'s time})^2 \therefore BM : CS :: (\text{time } M)^2 : (\text{time } S)^2$, but $BM : CS :: AB : AC \therefore AB : AC ::$ square of time of M : square of time of S.

Q. E. D.

J. C. Maxwell.
24th May /47.

Descartes' Ovals.

Definitions.

1. If a point move in a plane so that m times its distance from one point together with n times its distance from another be equal to a constant quantity it will describe a curve called an Oval of Descartes.
2. The two fixed points are called the foci the numbers expressed by m & n are called the powers of the foci, and that focus whose power is greatest is the greater focus, the other being the less.
3. The line joining the foci is called the Axis.

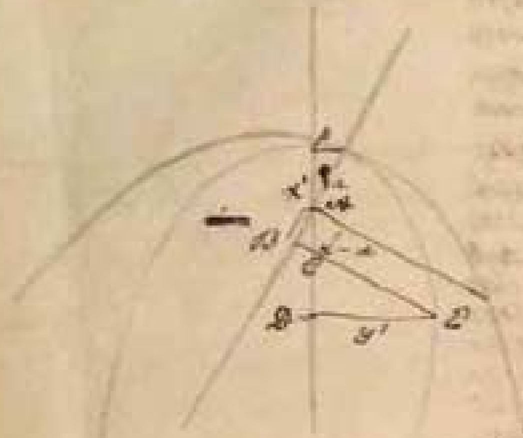
Proposition I.
Problem.

With given foci, given powers, and a given constant quantity, to describe an oval.

Let A & B be the foci, 3 & 2 their powers, and 7 the constant quantity, it is required to describe the Oval. As A & B [illegible]

Fig 1

[illegible]



$$\sqrt{a^2-p^2} : a :: y' : \frac{ay'}{\sqrt{a^2-p^2}} = CG$$

$$\frac{a^2-(a^2-p^2)}{a^2-p^2}\,y'^2 = \frac{p^2}{a^2-p^2}\,y'^2 = GB^2$$

$$AG = x' - a - \frac{p}{\sqrt{a^2-p^2}}\,y'$$

$$BG = \frac{p}{a}\left(x' - a - \frac{p}{\sqrt{a^2-p^2}}\,y'\right)$$

$$AB = \frac{\sqrt{a^2-p^2}}{a}\left(x' - a - \frac{p}{\sqrt{a^2-p^2}}\,y'\right)$$

$$hd\left(a + \frac{\sqrt{a^2-p^2}}{a}\left(x' - a - \frac{p}{a^2-p^2}\,y'\right)\right) =$$

$$\left[\left(\frac{a}{\sqrt{a^2-p^2}} - \frac{p^2}{a\sqrt{a^2-p^2}}\right)y' + \frac{p}{a}(x'-a)\right]^2$$

$$\frac{a^2-p^2}{a\sqrt{a^2-p^2}}$$

Propositions on the ~~circular~~ conical

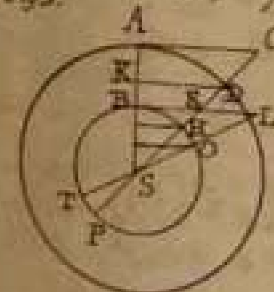
Pendulum.

Prop. 1. Theorem.

Centrifugal force increases directly as the radius & inversely as the square of the time of revolution.

For let A a body move from A to D & let AB be a uniform arc & the angle ABD very small and let p be the time of describing AD, & AB the direction in which it would move if free, then BD is the distracting force & the centrifugal force of B revolving also round S is EH & SE : SB :: SB : SH :: SB − SE : SE − SH :: BE : SH or EH : ED :: AD : AS or the centrifugal force increases directly as radius.

Fig. 1.



And let the times of revolution of ~~[illegible]~~ two bodies revolving round S with the same radius AL be as BL, [illegible] will get to O while the other gets to H & BE² = EF·EH & [illegible] = [illegible]·ED, but as BE & OL are very small BE² : BL² :: EH : OL or the centrifugal force is inversely as the square of the velocity.

Q. E. D.

J. C. Maxwell

26th May 1847.

$x\frac{dy}{dx} + y = f(x)$

$\therefore x\frac{dy}{dx} = 0 \therefore \frac{dy}{dx} = 0$

$y = f(x)$,

$\bar{y} = \frac{1}{x}\int_0^x f(x)\,dx$

$\frac{d}{dx}(\bar{x}\bar{y}) = f(x)$, (i.e.) $\bar{x}\frac{d\bar{y}}{dx} + \bar{y}\frac{d\bar{x}}{dx} = f(x)$

$\frac{d\bar{y}}{dx} = \frac{f(x) - \bar{y}\frac{d\bar{x}}{dx}}{x}$

$\frac{d\bar{y}}{dx} = \frac{1}{x}f(x) - \frac{1}{x^2}\int_0^x f(x)\,dx$

$\frac{d\bar{y}}{dx} = \frac{f(\bar{x}) - \bar{y}}{\bar{x}} = \frac{f(\bar{x}+h) - f(x)}{x} =$

$\frac{df(x)}{dx} + \frac{d^2f(x)}{dx^2}\frac{h^2}{1.2} + \ldots$

$= \frac{df(x)}{dx} + \frac{d^2f(x)}{dx^2}\frac{x^2 - f(\bar{y})}{1.2} + \ldots$

$\frac{f(\bar{x}) - f(x)}{x} = \frac{d\bar{y}}{dx}$

altern

A = 12 B's

∴ 13 B's amount to £156

(Solve by Algebra) B's amount £12

But B has only $\frac{3}{4}$ of what he had

∴ B had £16 & A £140

Summing ... sum of ... numbers ... the difference of the ...

their difference ...

In the centigrade thermometer the freezing point is zero & the boiling point 100°. In Fahrenheit's the freezing pt is 32° & the boiling point 212°. What is the temperature of Fahrenheit when centigrade stands at 75°.

180

.75

900

1260

135.00

32

167

3.5

$\frac{21 \times 5 \times 3.5}{20 \times 11 \times 1}$

75 : 100 :: ...

2. A's & B's Money together amount to £156, if B lose $\frac{1}{4}$ of his to A, A's will be 12 times B's. Required each by the aid of algebraic symbols.

3. If 1 be divided into any two parts the diff. of their squares is = the diff. of the parts themselves.

6 3   42 21

21

63

126

1.2

2.4

3.6

$y = ux$

$dy = x\,du + u\,dx$

$d^2y = x\,d^2u + 2\,du\,dx + u\,d^2x$

$6.45\,x = \frac{l}{.01}\log \quad 550.517 = \ldots \log$

$\frac{d^2y}{dx^2}$

dim = length = weight =

$$\int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\sqrt{dy^2+dx^2} = \alpha\frac{dy}{dx} =$$

$$\int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \alpha\frac{dy}{dx}$$

$$\int \sqrt{\left(\frac{dy}{dx}\right)^2+2+\left(\frac{dx}{dy}\right)^2}\,dx = \alpha\frac{dy}{dx}$$

$$\int \left(\frac{dy}{dx}+\frac{dx}{dy}\right)dx = \alpha\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}dx + \frac{dx}{dy}dx = \alpha\frac{d^2y}{dx^2}$$

$$dx^2dy^2 + dx^4 = \alpha^2 d^2y\,dy$$

$$dx^2dy^2 + dx^4 = \alpha^2 d^2y\,dy$$

$$p^2dx^4 + dx^4 = \alpha^2 p\,dx\,dx^2\,dp$$

$$(p^2+1)\,dx = \alpha^2 p\,dp$$

$$\frac{dx}{dp} = \frac{\alpha^2 p}{p^2+1}, \quad x = C + \alpha^2\tan^{-1}x$$

$$\int x^{m-1}dx(a+bx^n)^p = \frac{x^{m-n}(a+bx^n)^{p+1}}{b(pn+m)} - \frac{a(m-n)}{b(pn+m)}\cdot\int x^{m-n-1}dx(a+bx^n)^p$$

$$\int x^m dx(\lambda.x)^n = x^{m+1}\left\{\frac{(\lambda x)^n}{m+1} - \frac{n(\lambda.x)^{n-1}}{(m+1)^2} + \frac{n(n-1)(\lambda x)^{n-2}}{(m+1)^3} - \&c\right\} + C.$$

$$\int \frac{x^m \partial x}{(\lambda.x)^n} = \frac{-x^{m+1}}{(n-1)(\lambda x)^{n-1}} + \frac{m+1}{n-1}\times$$

$$\int \frac{x^m \partial x}{(\lambda.x)^{n-1}}$$

$$\int \frac{dx}{\sin.x} = \lambda(\tan\tfrac{1}{2}x) + C. = \tfrac{1}{2}\lambda.\frac{1-\cos x}{1+\cos x} + C.$$

$$\int \frac{dx}{\cos.x} = \lambda(\tan[\tfrac{1}{4}\pi + \tfrac{1}{2}x]) + C = \tfrac{1}{2}\lambda.\frac{1+\sin x}{1-\sin x} + C.$$

$$\int \frac{dx}{\tan.x} = \lambda.(\sin.x) + C.$$

$$\int dx\tan x = \lambda.\left(\frac{1}{\cos.x}\right) + C.$$

Solidity $= \pi\int y^2 dx.$

Surface $= \int 2\pi y\sqrt{dx^2 + dy^2}.$

$\delta = \epsilon^{-k}$ for $d\delta = \ldots d(\ldots k)$

$\sin(90-\varphi) = \cos\varphi \propto \frac{1}{\delta} = \alpha\epsilon^{y}$

~~$\tan\varphi = \frac{dy}{dx}$~~

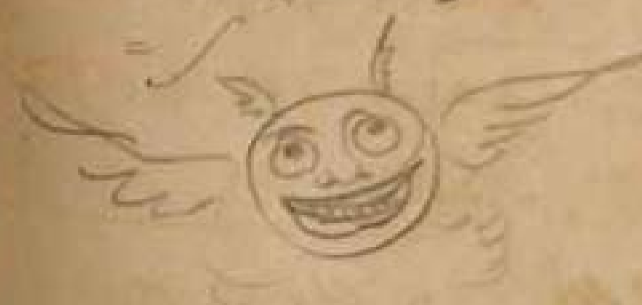
but $\cos\varphi = \frac{dx}{\sqrt{dy^2+dx^2}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$

$\therefore \alpha\epsilon^y = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$ $\therefore$

$$\frac{dy}{dx} = \sqrt{[\alpha\epsilon^y]^{-2} - 1}$$

$$\therefore x = \int_y \left(\frac{1-\alpha^2\epsilon^{2y}}{\alpha^2\epsilon^{2y}}\right)^{\frac{1}{2}}$$

$= \int$



Let the roots of the denominator assumed = 0 be $x, \alpha', \alpha'', \alpha'''$ &c then assume the fraction =

$$\frac{N\,dx}{x+a} + \frac{N'\,dx}{x+\alpha'} + \frac{N''\,dx}{x+\alpha''} + \text{\&c}$$

By adding these and assuming the coefficients of the powers of x equal we find N, N', N'' &c & then the decomposed fractions is integrated by the preceding formulae.

$$\int\frac{(Kx+L)\,dx}{x^2+2ax+a^2+\beta^2} = C + K\lambda.\sqrt{(x^2+2ax+a^2+\beta^2)} + \frac{L-Ka}{\beta}.\tan^{-1}\frac{x+a}{\beta}.$$

$$\int\frac{(Kx+L)\,dx}{(x^2+2ax+a^2+\beta^2)^q} = \frac{K}{2}\frac{(x^2+2ax+a^2+\beta^2)^{1-q}}{1-q} + (L-Ka)\left(\frac{1}{(2q-2)\beta^2}\cdot\frac{x+a}{(x^2+2ax+a^2+\beta^2)^{q-1}} + \frac{2q-3}{(2q-2)\beta^2}\cdot\int\frac{d(x+a)}{(x^2+2ax+a^2+\beta^2)^{q-1}}\right).$$

$$\int\frac{a\,dx}{x^2-a^2} = \lambda\sqrt{\frac{x-a}{x+a}} + C.$$

$$\int\frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\left(\lambda\left[\frac{b}{2\sqrt{c}} + x\sqrt{c} + \sqrt{a+bx+cx^2}\right]\right) + C$$

$$= \frac{1}{\sqrt{c}}.\lambda\left(b + 2cx + 2\sqrt{c}\,\sqrt{a+bx+cx^2}\right)$$

$$\int\frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}}\cos^{-1}\frac{b-2cx}{\sqrt{b^2+4ac}} + C = -\frac{1}{\sqrt{c}}\sin^{-1}\frac{2cx-b}{\sqrt{b^2+4ac}} + C.$$

$$\int\frac{x^n\,dx}{\sqrt{a+bx+cx^2}} = \frac{1}{cn}x^{n-1}\sqrt{a+bx+cx^2} - \frac{2n-1}{2n}\cdot\frac{b}{c}\int\frac{x^{n-1}\,dx}{\sqrt{a+bx+cx^2}} - \frac{n-1}{n}\cdot\frac{a}{c}\int\frac{x^{n-2}\,dx}{\sqrt{a+bx+cx^2}},$$ when n is a whole number.

$$\int\frac{dx}{x^n\sqrt{a+bx+cx^2}} = \frac{-1}{(n-1)a}\cdot\frac{\sqrt{a+bx+cx^2}}{x^{n-1}} - \frac{2n-3}{2n-2}\cdot\frac{b}{a}\int\frac{dx}{x^{n-1}\sqrt{a+bx+cx^2}} - \frac{n-2}{n-1}\cdot\frac{c}{a}\int\frac{dx}{x^{n-2}\sqrt{a+bx+cx^2}}.$$

$$\int\frac{dx}{x\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{a}}\lambda\frac{[illegible]}{x} + C$$

$$\int\frac{dx}{x\sqrt{-a+bx+cx^2}} = \frac{1}{\sqrt{a}}\cos^{-1}\frac{2a-bx}{x\sqrt{b^2+4ac}} + C.$$

Let $\cos\varphi = \sqrt{\frac{x^2-\beta^2}{\alpha^2-\beta^2}}$, & $\cos\psi = \frac{\beta}{\alpha}$ [illegible]

$$\int\frac{dx}{x}\sqrt{\frac{\alpha^2-x^2}{x^2-\beta^2}} = \varphi + \psi + \text{constant}$$

$\frac{dt}{} \propto \frac{dz}{}$

$\omega \propto \tan$

$\int dt \sqrt{dx^2+dy^2} \propto$

$\int \sqrt{dx^2+dy^2} = m \frac{dy}{dx}\frac{d^2y}{dx^2}$

$\int dx^2 + \int dy^2 = m\, dx^2 \frac{dy}{dx} \frac{d^2y}{dx^2}$

$\int + \int \left(\frac{dy}{dx}\right)^2 = m \frac{dy}{dx} \frac{d^2y}{dx^2}$

$\int + \int p^2 = m p \frac{dp}{dx}$

$dx = \frac{m p\, dp}{\int + \int p^2}$

$1 + p^2 =$

$\frac{1}{2} \log(1+p^2)$

$\int a x^m dx = \frac{a}{m+1} x^{m+1} + C.$

$\int a x^n dx = \frac{a(x^{n+1} - b^{n+1})}{n+1}.$

if the function vanishes when $x = b$

$\int \frac{a\,\partial x}{x} = a(\lambda x - \lambda b) = a \lambda x + C.$

$\int u\,dv = uv - \int v\,du.$

$\int u \frac{dv}{v^2} = -\frac{u}{v} + \int \frac{du}{v}$

$\int a X \partial x = a \int X \partial x.$

$\int (ax+b)^m \partial x = \frac{(ax+b)^{m+1}}{a(m+1)} + C.$

$\int (ax^n+b)^m x^{n-1} dx = \frac{(ax^n+b)^{m+1}}{na(m+1)} + C.$

$\int \frac{A x^m dx}{(ax+b)^n}$ making $z = ax + b$, $x = \frac{z-b}{a}$

$dx = \frac{dz}{a}$ therefore the integral is $= \int \frac{A(z-b)^m dz}{a^{m+1} z^n}.$

developing $(z-b)^m$ and transforming into a series each term can be integrated by the two first formulae

All rational fractional differentials may be reduced to the form

$\frac{(A x^{n-1} + B x^{n-2} + \cdots + T) dx}{x^n + A' x^{n-1} + B' x^{n-2} \cdots + T'}$

Evolutes & Involutes of Curves.

$$r = \frac{ds}{d\varepsilon}$$

$$r = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y}$$

$$r = -\frac{s^3}{y^2}\cdot\frac{dx^2}{dy^2}$$

} y is a function of x.

$$r = p + \frac{d^2p}{d\varepsilon^2}.$$

$$\alpha = x - \frac{dy}{dx}\cdot\frac{dx^2 + dy^2}{d^2y}, \quad \beta = y + \frac{dx^2 + dy^2}{d^2y}.$$

where also y is a function of x.

Let the curve be the Parabola

$$r = \frac{4s^3}{c^2} = \frac{s^3}{(\frac{1}{2}c)^2} = \frac{(\text{normal})^3}{(\text{semi-param})^2}$$

$\alpha = 3x + \frac{1}{2}c$, $\beta = \frac{-4y^3}{c^2}$ ∴ the equation to the Evolute is $\beta^2 = \frac{16}{27c}\left(\alpha - \frac{c}{2}\right)^3$

The equation of that to the Ellipse is

$$\left(\frac{ax}{c^2}\right)^{\frac{2}{3}} + \left(\frac{by}{c^2}\right)^{\frac{2}{3}} = 1.$$

That of the cycloid is a^n equal^n inverted cycloid & its radius generating semicycloids and its

r is $\sqrt{8(1 + \cos\theta)}$.

Contact of Curves.

If two curves having a common abscissa & ordinate have their 1st, 2nd, 3rd &c. successively greater number of the orders of the ordinates = each other, neither curve which has not the same properties can pass between them.

Equations containing two variables,

$$du = \frac{du}{dt}dt + \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz$$

where $\frac{du}{dt}$ implies that u has been differentiated supposing only t variable & so on.

In this case $u = f(t, x, y, z)$.

Omnia, Satani fabri ..., ...

$\therefore X(a) = -1 \therefore x = c^{-1} = \frac{1}{c}$

Substituting in $\frac{d^2y}{dx^2}\left(\frac{1}{c}\right)^{\frac{1}{c}} \cdot c$ a positive quantity therefore $\left(\frac{1}{c}\right)^{\frac{1}{c}}$ is the minimum value of the function.

I. Determination of Tangents to Curves.

The subtangent $= y \div \frac{dy}{dx} = y\frac{dx}{dy}$.

The subnormal $= y \times \frac{dy}{dx} = y\frac{dy}{dx}$.

The Tangent $= y\sqrt{1 + \frac{dx^2}{dy^2}}$.

The Normal $= y\sqrt{1 + \frac{dy^2}{dx^2}}$

To draw a tangent to the cycloid at the point P.

$AOD = v,\ OC = a.$

$x =$ ~~[illegible]~~ $a(1 - \cos v)$,

$y = a(v + \sin v)$.

$dx = a \sin v,\ dy = a\,dv(1 + \cos v)$

$\frac{dx}{dy} = \frac{\sin v}{1 + \cos v}$. Subtan. $QT = y\frac{dx}{dy}$

$= \frac{(v + \sin v)\sin v}{1 + \cos v} = \frac{PQ \cdot QD}{CQ} \therefore$

$CQ : DQ :: PQ : QT :: DQ : QA \therefore$

PQT, DQA are similar and $PT \parallel DA$

II. Curves expressed by a polar equation.

Subtangent $TA = \frac{dv}{dr}r^2$.

Subnormal $AD = \frac{dr}{dv}$.

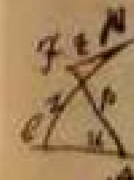
The differential of a curvilinear area is = the product of the ordinate & the differential of the abscissa. $ds = y\,dx$.

$= \frac{1}{2}$ the square of the revolving radius × differential of the angle which it makes with the axis. $ds = \frac{1}{2}r^2 dv$.

The square of the differential of an arc of a curve is = the sum of the squares of the differentials of its rectangular co-ordinates. $dz = \sqrt{dx^2 + dy^2}$.

When expressed by a polar equation the length of the curve is expressed thus

$$dz = \sqrt{r^2 dv^2 + dr^2}.$$



$d(z \mp t) = p\,du$

but $t = \pm\frac{dp}{du}$

$\therefore dz = p\,du + \frac{d^2p}{du^2}du.$

Of the greatest and least values of a Function.

When any value of a function is greater or less than both the nearest preceding & succeeding values it is then said to be a maximum or minimum.

It is easily proved that in a series like $ph + qh^2 + rh^3 + \&c$ we may give such a value to x that the terms after ph are so small as almost to disappear. Hence h may have such a value that the amount of all the terms beginning with an assumed term shall have the sign of that term. Let y be any function. Substituting in the function $(x+h)$ & $(x-h)$ for x we obtain the following rule;

Make the differential coefficient $\frac{dy}{dx} = 0$, find the resulting value of x & substitute it in $\frac{d^2y}{dx^2}$; if the result be negative the function is not a minimum but if ~~[illegible]~~ positive it is a minimum & If it $= 0$ put $\frac{d^3y}{dx^3} = 0$ find x & substitute in $\frac{d^4y}{dx^4}$ and then as before.

Problems.

1) To determine whether $u = 2ax - x^2$ has max or min values;
Here $\frac{du}{dx} = (2a - 2x)$ $\therefore x = a$.
The second diff. coeff. $\frac{d^2u}{dx^2} = -2$ $\therefore$ the value of u a maximum when $x = a$ (which $u = a^2$)

2) ~~$u = x^2 + 3x + 2$~~ $\therefore \frac{du}{dx} = 2x + 3$ $\therefore x = -\frac{3}{2}$
Again $\frac{d^2u}{dx^2} = 2$ $\therefore u = -\frac{1}{4}$ is the minimum value.

3) $u = x^3 - 15x^2 + 56x - 60$ $\therefore$
$\frac{du}{dx} = 3x^2 - 30x + 56$, $\frac{d^2u}{dx^2} = 6x - 30$.
$\therefore x = \frac{15 \pm \sqrt{57}}{3}$ & $\frac{d^2u}{dx^2} = \pm 2\sqrt{57}$ $\therefore$
The minimum value is $-30 - \frac{38}{9}\sqrt{57}$
& the maximum $= -30 + \frac{38}{9}\sqrt{57}$.

4) $u = x^5 - 5x^4 + 5x^3 + 1$.
The 4 values of x are 3, 1, 0, 0.
When $x = 3$, $u = -26$ a minimum.
$x = 1$, $u = 2$ a maximum.
When $x = 0$ $\frac{d^2u}{dx^2}$ vanishes $\therefore$ there is neither max. nor minim. in this case.

5) $y = x^x$
$\frac{dy}{dx} = x^x(1 + \lambda(x))$
$\therefore \frac{d^2y}{dx^2} = x^x\left(\frac{1}{x} + \{1 + \lambda(x)\}^2\right)$.
x^x cannot $= 0$ $\therefore (1 + \lambda(x)) = 0$.

2) Let the fraction be $\frac{ax^2+ac^2-2acx}{bx^2-2bcx+bc^2}=\frac{P}{Q}$

which becomes $=\frac{0}{0}$, when $x=c$.

$\frac{dP}{dx}=2ax-2ac=P'$, $\frac{dQ}{dx}=2bx-2bc=Q'$ ∴ $\frac{P'}{Q'}=\frac{ax-ac}{bx-bc}$ which when $x=c$ also becomes $=\frac{0}{0}$ ∴ proceeding as before

$\frac{d^2P}{dx^2}=2a=P''$, $\frac{d^2Q}{dx^2}=2b=Q''$ ∴ $\frac{P''}{Q''}=\frac{a}{b}$ the real value of the above fraction when $x=c$.

3) $\frac{a^x-b^x}{x}$ becomes $=\frac{0}{0}$ when $x=0$.

$\frac{dP}{dx}=\lambda(a)\,a^x-\lambda(b)\,b^x=P'$, $\frac{dQ}{dx}=1=Q'$. When $x=0$, $\frac{P'}{Q'}$ becomes $=\lambda(a)-\lambda(b)=\lambda\left(\frac{a}{b}\right)$.

4) $\frac{1-\sin x+\cos x}{\sin x+\cos x-1}$ which when $x=\frac{\pi}{2}$ is $=\frac{0}{0}$.

$\frac{dP}{dx}=-\cos x-\sin x=P'$, $\frac{dQ}{dx}=\cos x-\sin x=Q'$ ∴ $\frac{P'}{Q'}=\frac{-\cos x-\sin x}{\cos x-\sin x}$. When $x=\frac{\pi}{2}$, $\cos x=0$, $\sin x=1$ ∴ $\frac{P'}{Q'}=\frac{-1}{-1}=1=$ the value of the proposed fraction when $x=\frac{\pi}{2}$.

5) $\frac{a^{x-n}-x^{x-n}}{a-x}$, find its value when $x=a$.

$P'=\frac{dP}{dx}=-x^{x-n}\left\{\lambda(x)+\frac{x-n}{x}\right\}$, $Q'=\frac{dQ}{dx}=-1$ ∴ $\frac{P'}{Q'}=x^{x-n}\left\{\lambda(x)+\frac{x-n}{x}\right\}$

when $x=a$ this becomes $=$

$$a^{a-n}\left\{\lambda(a)+\frac{a-n}{a}\right\}$$

A rule which extends even to those cases in which Taylor's Theorem fails.

Take the first term of each of the series which express the developement of the Num. & Denr. when $x=(a+h)$, reduce the resulting fraction to its lowest terms & then make $h=0$.

$\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$ whose value cannot be found by differentiation becomes by this method when $x=a$ $\frac{(2ah+h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}}=(2a+h)^{\frac{3}{2}}$ which when $h=0$ becomes $2a^{\frac{3}{2}}$ the value required.

$\frac{x^3+4ax^2+7a^2x-2a^3-2a^2\sqrt{2ax-a^2}}{x^2-2ax-a^2+2a\sqrt{2ax-x^2}}$ which would require a troublesome differentiation becomes by this method $=-5a$.

Vanishing Fractions.

When the numerator & denominator of a fraction are functions which disappear at once, by giving a particular value to their variable they are called Vanishing Fractions. Thus, $\frac{x^2-a^2}{x-a}$ becomes (when $x = a$) $\frac{0}{0}$ although we know that its value is then $= 2a$. Here the reason is obvious viz. $(x-a) = 0$ is a factor in the numerator & is also the denominator ∴ both become $= 0$.

Sometimes the common factor cannot be so easily discovered, thus, $\frac{1-\sin x+\cos x}{\sin x+\cos x-1}$ becomes $\frac{0}{0}$ when $x = \frac{\pi}{2}$. For its true value in this case see one of the following examples.

Let $\frac{P}{Q}$ denote a fraction the terms of which vanish when $x = a$ (a given quantity). Suppose now that x becomes $= (x+h)$ then by Taylor's Theorem the fraction becomes

$$\frac{P+\frac{dP}{dx}h+\frac{d^2P}{dx^2}\frac{h^2}{1\cdot2}+\frac{d^3P}{dx^3}\frac{h^3}{1\cdot2\cdot3}+\&c}{Q+\frac{dQ}{dx}h+\frac{d^2Q}{dx^2}\frac{h^2}{1\cdot2}+\frac{d^3Q}{dx^3}\frac{h^3}{1\cdot2\cdot3}+\&c}$$

By hypothesis when $x = a$, P & Q both vanish ∴ leaving them out & putting P', P'' &c to represent the differential coefficients of P in the numerator & Q', Q'' &c those of Q in the denominator & dividing by h the fraction becomes

$$\frac{P'+\frac{1}{2}P''h+\frac{1}{6}P'''h^2+\&c}{Q'+\frac{1}{2}Q''h+\frac{1}{6}Q'''h^2+\&c}$$

which when $h = 0$, becomes $= \frac{P'}{Q'}$ the true value of the fraction when x is $= a$. If it becomes $= \frac{0}{0}$ take $\frac{P''}{Q''}$ & treat it in the same way till a fraction is found which does not become $= \frac{0}{0}$; this is the real value of the vanishing fraction.

Examples.

1) The sum of the series to n terms $1 + x + x^2 + x^3 + \&c$ is $\frac{x^n-1}{x-1}$ find this when $x = 1$.

Here $P = x^n - 1$, $Q = x - 1$ ∴ $\frac{dP}{dx} = nx^{n-1} = P'$; $\frac{dQ}{dx} = 1 = Q'$ ∴ $\frac{P'}{Q'} = nx^{n-1}$ & when $x = 1$ the sum is n as is otherwise manifest

Maclaurin's Theorem.

We have seen that when $f(x) = u$,

(A) $f(x+h) = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{2} + \frac{d^3u}{dx^3}\frac{h^3}{2\cdot 3} + \&c$

Let $x = 0$, and suppose that $f(x) = u$ becomes U, that $\frac{du}{dx}$ becomes U', $\frac{d^2u}{dx^2}$ becomes U'', &c

Substituting in (A) and putting x for h,

$$f(x) = U + U'x + U''\frac{x^2}{1\cdot 2} + U'''\frac{x^3}{1\cdot 2\cdot 3} + \&c$$

From this it appears that if u can be expand into, $u = A + Bx + Cx^2 + Dx^3 + \&c$ where A, B &c are constants then U being the value of u when $x = 0$, & U', U'' &c being the values of $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$ &c found on that hypothesis we have $A = U$, $B = U'$ &c and the theorem becomes,

$$u = U + U'x + U''\frac{x^2}{2} + U'''\frac{x^3}{2\cdot 3} + \&c$$

This is Maclaurin's Theorem —

Differentiation of Equations containing two Variables.

By solving the equation for one of them we may easily differentiate it by the foregoing rules, but in general we cannot thus separate the variables.

We may represent an equation compounded of two variables by $f(x, y)$ or $f(x, X)$ where $X = y$ is an equation containing only x & constants. Let $f(x, X) = u$. Let x become $(x + h)$

$$u' = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{\text{[illegible]}} + \&c.$$

$f(x, y) = u = 0$ must hold good whatever be the value of x $\therefore \frac{du}{dx} = 0$, $\frac{d^2u}{dx^2} = 0$, and so on.

Let us take for example the equation $y^2 + x^2 - a^2 = 0$, then differentiating we have

$$\frac{du}{dx} = 2y\frac{dy}{dx} + 2x = 0 \therefore \frac{dy}{dx} = -\frac{x}{y}.$$

If $p = \frac{dy}{dx}$ we have

$$\frac{du}{dx} = 2yp + 2x,$$ Differentiating & dividing by dx we have

$$\frac{d^2u}{dx^2} = 2\left(y\frac{dp}{dx} + p\frac{dy}{dx} + 1\right) = 0$$

Hence $\frac{d^2y}{dx^2} = -\frac{1}{y}\left(\frac{dy^2}{dx^2} + 1\right)$ or since $\frac{dy}{dx} = -\frac{x}{y}$, we get

$$\frac{d^2y}{dx^2} = -\frac{x^2}{x^3} - \frac{1}{y}.$$

The following is the substance of Lagrange's investigation

let $f(x)$ be any function of x, when x becomes $(x+h)$

$$f(x+h) = f(x) + ph + qh^2 + rh^3 + \&c \quad (A)$$

let x now become $x+k$

$f(x+h)$ becomes $f(x+h+k)$.

To find what $f(x) + ph + qh^2$ &c becomes when $x+k$ is substituted for x we [redacted] have two methods 1st by substituting $(h+k)$ for h wherever it occurs. 2nd by substituting $(x+k)$ for x in the functions p, q, r &c

By the first process $f(x+h+k) =$

$$= \begin{matrix} u + ph + qh^2 + rh^3 + \\ + pk + 2qhk + 3rh^2k + \\ + qk^2 + 3rhk^2 + \\ + rk^3 + \\ + \end{matrix} \quad \&c \quad (B)$$

In employing the second we must consider that when x becomes $= x+k$, then u becomes $u + pk + qk^2 + rk^3 + \&c$

The supposition that x changes its value to $x+k$ leads to corresponding changes in p, q, r &c so that

p becomes $p + p'k + p''k^2 + p'''k^3$ &c

$q + q'k + q''k^2 + q'''k^3$ &c

Substituting now in equation (A) these last values we have $f(x+h+k) =$

$$= \begin{matrix} u + pk + qk^2 + rk^3 + sk^4 + \\ + ph + p'hk + p''hk^2 + p'''hk^3 + \\ + qh^2 + q'h^2k + q''h^2k^2 + \\ + rh^3 + r'h^3k + \&c \\ + sh^4 + \\ + \end{matrix} \quad (C)$$

To make Equations (B) & (C) identical we must have $q = \frac{p'}{2}$, $r = \frac{q'}{3}$, $s = \frac{r'}{4}$ &c

Now by the definition of a differential pdx = diff of u where p is the coefficient of h in $u + ph + qk^2$ &c $\therefore p = \frac{du}{dx}$. Similarly $q' = \frac{dq}{dx}$, $r' = \frac{dr}{dx}$ and so on.

Hence $p = \frac{du}{dx}$

$$q = \frac{1}{2}p' = \frac{1}{2}\frac{dp}{dx} = \frac{1}{1.2}\frac{d^2u}{dx^2}.$$

$$r = \frac{1}{3}q' = \frac{1}{3}\frac{dq}{dx} = \frac{1}{1.2.3}\frac{d^3u}{dx^3}.$$

And so on $\therefore$ substituting in (A)

$$f(x+h) =$$

$$= u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \&c$$

where $u = f(x)$ & this is Taylor's theorem.

$$dz = n\,du\,(\cos nu \cos^n u - \sin nu \sin u \cos^{n-1} u) =$$

$$= n\,du \cos^{n-1} u\,(\cos nu \cdot \cos u - \sin nu \cdot \sin u) =$$

$$= n\,du \cos^{n-1} u \cdot \cos(n+1)u, \;\&,$$

$$\frac{dz}{dx} = n \cos(n+1)u \cdot \cos^{n+1} u.$$

Successive Differentiation — Notation —

When a function of a variable is differentiated it produces a new function viz. its differential coefficient this may again be differentiated & so on until some differential coefficient comes out = 0.

E.g. $u = x^n$, $\frac{du}{dx} = nx^{n-1} = p$,

$\frac{dp}{dx} = n(n-1)x^{n-2} = q$, $\frac{dq}{dx} =$

$= n(n-1)(n-2)x^{n-3}$ and so on —

$p = \frac{du}{dx}$ & $q = \frac{dp}{dx} = \frac{d\left(\frac{du}{dx}\right)}{dx}$. Since dx is invariable $d\left(\frac{du}{dx}\right) = \frac{d^2u}{dx}$ ∴ $p = \frac{du}{dx}$,

$q = \frac{dp}{dx} = \frac{d^2u}{dx^2}$, $r = \frac{d^3u}{dx^3}$.

d^2u means that the process of differentiation has been performed on u & the other dx^2 means the square of the first differential of x or $(dx)^2$

We shall now give a few examples of this process.

1) $u = a^x$, $\frac{du}{dx} = \lambda \cdot a \cdot a^x$, $\frac{d^2u}{dx^2} = (\lambda a)^2$ [illegible] a^x,

$\frac{d^3u}{dx^3} = (\lambda a)^3 a^x$ & so on.

2) $u = \log x$ or $x = a^u$ then $\frac{du}{dx} = \frac{M}{x}$

$\frac{d^2u}{dx^2} = -\frac{M}{x^2}$, $\frac{d^3u}{dx^3} = -\frac{1 \cdot 2M}{x^3}$, $\frac{d^4u}{dx^4} = \frac{1 \cdot 2 \cdot 3M}{x^4}$.

3) $u = \sin x$, $\frac{du}{dx} = \cos x$, $\frac{d^2u}{dx^2} = -\sin x$

$\frac{d^3u}{dx^3} = -\cos x$, $\frac{d^4u}{dx^4} = \sin x$ &c

4) $u = \cos x$, $\frac{du}{dx} = -\sin x$, $\frac{d^2u}{dx^2} = -\cos x$,

$\frac{d^3u}{dx^3} = \sin x$, $\frac{d^4u}{dx^4} = \cos x$ &c

5) $u = \frac{a^2}{a^2+x^2}$; $\frac{du}{dx} = \frac{-2a^2x}{(a^2+x^2)^2}$,

$$\frac{d^2u}{dx^2} = \frac{-2a^4 + 6a^2x^2}{(a^2+x^2)^3}$$

$$\frac{d^3u}{dx^3} = \;\&c$$

Taylor's Theorem

It may easily be demonstrated [*] that if in any function of x we substitute $x+h$ for x that function will be expanded into the form $f(x) + ph + qh^2 + rh^3 + sh^4$ &c. To find the relation between $f(x)$ and p, q, r &c derived from it is the object of Taylor's Theorem.

[*] as already [illegible]

Examples of the Differentiation of Complex Functions.

1.) $u = x^y \quad \therefore \lambda u = y \lambda x.$

Let $v = \lambda u$ & $z = \lambda x$ then $v = yz$

& $dv = y dz + z dy.$

Now $dv = d(\lambda u) = \frac{du}{u}$, similarly $dz = \frac{dx}{x}.$

$\therefore \frac{du}{u} = \frac{y dx}{x} + \lambda x dy,$

& $du = \left(u \left\{\frac{y}{x} dx + \lambda x\, dy\right\}\right) = \left(x^y \left\{\frac{y}{x} dx + \lambda x dy\right\}\right) = x^{y-1} y dx + x^y \lambda x dy.$

2) $u = x^x.$ By last example,

$$du = x^x\{1 + \lambda x\} dx.$$

3) Let $u = a^{b^x}$; put $b^x = y$ then $u = a^y$ &

$du = \lambda a . a^y . dy$ also $dy = \lambda b . b^x . dx \therefore$

$$du = \lambda a \lambda b . a^{b^x} dx.$$

We shall in the following suppose the Logarithms to be those of Napier.

4) $u = \lambda \left\{\frac{x}{\sqrt{a^2 + x^2}}\right\} = \lambda z \quad \therefore du = \frac{dz}{z}$ but

$$dz = \frac{\sqrt{a^2 + x^2}\, dx - \frac{x^2 dx}{\sqrt{a^2 + x^2}}}{a^2 + x^2} = \frac{a^2 dx}{(a^2 + x^2)^{\frac{3}{2}}}$$

$$\therefore du = \frac{a^2 dx}{x(a^2 + x^2)}$$

5) Let $u = \lambda \left\{\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}\right\} = \lambda\left(\frac{y}{z}\right) =$

$= \lambda(y) - \lambda(z) \quad \therefore du = \frac{dy}{y} - \frac{dz}{z}.$

Now $dy = \frac{dx}{2\sqrt{1+x}} - \frac{dx}{2\sqrt{1-x}} = \frac{-z\, dx}{2\sqrt{1-x^2}}.$

$= \frac{-dx}{2\sqrt{1-x^2}}\left\{\sqrt{1+x} - \sqrt{1-x}\right\}$

$dz = \frac{dx}{2\sqrt{1+x}} + \frac{dx}{2\sqrt{1-x}} = \frac{dx}{2\sqrt{1-x^2}}\left\{\sqrt{1+x} + \sqrt{1-x}\right\}$

$= \frac{y\, dx}{2\sqrt{1-x^2}} \therefore \frac{dy}{y} - \frac{dz}{z} = \frac{-z^2 x}{2y\sqrt{1-x^2}} - \frac{y^2 x}{2z\sqrt{1-x^2}}$

$= \frac{-(y^2 + z^2) dx}{2yz\sqrt{1-x^2}}$ and observing that $y^2 + z^2 = 4$

$yz = 2x$ we find at last $du = \frac{-dx}{x\sqrt{1-x^2}}.$

In the last two we shall merely state the results.

6) $u = \lambda\left\{x - \sqrt{1+x^2}\right\}, \; du = \frac{dx}{\sqrt{1+x^2}}.$

7) $u = \lambda \left\{\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2} - x}\right\} \quad du = \frac{dx}{\sqrt{1+x^2}}$

We shall now give a few examples combining logarith. & angular functions.

1) Let $u = \lambda . \sin x.$

$du = \frac{dx \cos x}{\sin x} = dx \tan x.$

2) Let $u = \lambda \tan x.$

$du = \frac{dx \sec^2 x}{\tan x} = \frac{dx}{\sin x \cos x} = \frac{2 dx}{\sin 2x}.$

Examples of complex angular functions.

1) Let $u = \tan^{-1} x$ & let $z = \cos nu \cos^n u.$

Required $\frac{dz}{dx}.$

but $du = \frac{dx}{\sec^2 u} = dx \cos^2 u.$

$dz = -n du(\sin nu . \cos^n u + \cos nu . \sin u . \cos^{n-1} u) =$

$= -n du \cos^{n-1} u(\sin nu . \cos u + \cos nu . \sin u) =$

$= -n du . \cos^{n-1} u . \sin(n+1) u.$

& $\frac{dz}{dx} = -n \sin(n+1) u . \cos^{n+1} u.$

2) Next to find the value of $\frac{dz}{dx}$, u being the same function of x as before & $z = \sin nu \ldots$

Let now $a = 1 + c$ & for (binomial Theorem),

$a^h = (1+c)^h = 1 + hc + \frac{h(h-1)}{1\cdot 2}c^2 + \frac{h(h-1)(h-2)}{1\cdot 2\cdot 3}c^3$ &c.

$\therefore \frac{a^h - 1}{h} = c + \frac{h-1}{2}c^2 + \frac{(h-1)(h-2)}{2\cdot 3}c^3 +$ &c.

Now supposing h to decrease continually the ~~left~~ right side becomes

$c - \frac{1}{2}c^2 + \frac{(-1)(-2)}{2\cdot 3}c^3 + \frac{(-1)(-2)(-3)}{2\cdot 3\cdot 4}c^4 +$ &c

$= c - \frac{1}{2}c^2 + \frac{1}{3}c^3 - \frac{1}{4}c^4 +$ &c. but $c = (a-1)$

$\therefore$ this series is = Napiers log of a.

$\therefore$ limit of $\frac{a^h - 1}{h}$ = Nap. log. a, hence

$\frac{du}{dx}$ = limit $\frac{u' - u}{h} = a^x$(Nap log a), and $\therefore$

$du = a^x$(Nap log a)dx.

N.B. Henceforward the Nap. log. of a will be expressed by $\lambda . a$ while in any other system it is log a

when $x = a^u$ to find du.

~~By last prop. dx = (a^u) x (Nap~~

By last prop. $dx = x(\lambda . a)du$ $\therefore$

$du = \frac{1}{\lambda . a} . \frac{dx}{x} = \frac{M dx}{x}$.

Let $u = \sin x$ & $u' = \sin(x + h)$ then (X)

$u' - u = 2\cos(x + \frac{h}{2})\sin\frac{h}{2}$ $\therefore$

$\frac{u' - u}{h} = \cos(x + \frac{h}{2})\frac{\sin\frac{h}{2}}{\frac{h}{2}}$. If h were

[illegible] limit of this will be $\cos x$ (1).

$\therefore \frac{du}{dx} = \frac{u' - u}{h} = \cos x$ $\therefore$ $du = dx \cos x$.

Let $u = \cos x$ & $u' = \cos(x + h)$ then $(u' - u)$

$= -2\sin(x + \frac{h}{2})\sin\frac{h}{2}$, finding now the limits as before $\frac{du}{dx}$ = limit $\frac{u' - u}{h} = -\sin x$ $\therefore$

$du = -dx . \sin x$.

Let $u = \tan x$, then $du = \frac{d(\sin x)\cos x - d(\cos x)\sin x}{\cos^2 x}$

= (by last prop) $= \frac{(\cos^2 x + \sin^2 x)dx}{\cos^2 x} = \frac{dx}{\cos^2 x}$.

Hence $du = dx . \sec^2 x = dx(1 + \tan^2 x)$

$u = \cot x$.

$du = \frac{-dx}{\sin^2 x} = -dx . \operatorname{cosec}^2 x = -dx(1 + \cot^2 x)$

Let $u = \sec x = \frac{1}{\cos x}$.

$du = \frac{dx \sin x}{\cos^2 x} = dx \tan x \sec x$.

Let $u = \operatorname{cosec} x = \frac{1}{\sin x}$

$du = -\frac{dx \cos x}{\sin^2 x} = dx \cot x \operatorname{cosec} x$.

Let $u = \sin^{-1} x$ then $x = \sin u$ and by the former prop. $dx = du \cos u = du\sqrt{1 - x^2}$

$\therefore du = \frac{dx}{\sqrt{1 - x^2}}$

In the same way we find that when

$u = \cos^{-1} x$, $du = \frac{-dx}{\sqrt{1 - x^2}}$

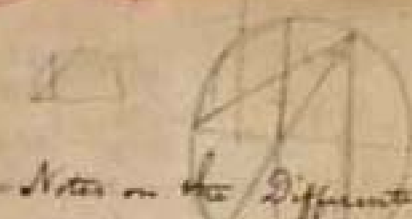
$u = \tan^{-1} x$, $du = \frac{dx}{1 + x^2}$.

$u = \cot^{-1} x$, $du = \frac{-dx}{1 + x^2}$

$u = \sec^{-1} x$, $du = \frac{dx}{x\sqrt{x^2 - 1}}$

$u = \operatorname{cosec}^{-1} x$. $du = \frac{-dx}{x\sqrt{x^2 - 1}}$

where $u = \sin^{-1} x$ means u is the arc of which x is the sine.



Notes on the Differential & Integral Calculus.

Generally

$$d(rstv) = rstv\left(\frac{dr}{r} + \frac{ds}{s} + \frac{dt}{t} + \frac{dv}{v}\right).$$

Let $u = \frac{r}{s}$ then $us = r$ & $dr = u\,ds + s\,du$

substituting for u its value $\frac{r}{s}$ we have

$$dr = \frac{r\,ds}{s} + s\,du \therefore du = \frac{s\,dr - r\,ds}{s^2}$$

or generally when $u = \frac{rst}{vy}$,

$$\frac{du}{u} = \frac{dr}{r} + \frac{ds}{s} + \frac{dt}{t} - \frac{dv}{v} - \frac{dy}{y}.$$

$$d(y^n) = ny^{n-1}dy \text{ or } d(ax^n) = nax^{n-1}dx.$$

$$d(\sqrt{y}) = \frac{dy}{2\sqrt{y}}$$

When $u = a^x$ to find the value of du.

$u = a^x$, $u' = a^{x+h} = a^x a^h$, $\therefore u' - u = a^x(a^h - 1)$.

& the limit of the ratio $\frac{u'-u}{h} = a^x \cdot \frac{a^h - 1}{h}$.

$$\int dy = \frac{1}{2}\int \epsilon^{\frac{1}{m(a+x)}+n}dx - \int \epsilon^{-\left(\frac{1}{m(a+x)}+n\right)}dx$$

$$y = \frac{\ldots}{(a+x)\epsilon^{\frac{1}{m(a+x)}+n}} + \frac{\epsilon^n}{m}\int_x \frac{\epsilon^{\frac{1}{m(a+x)}}}{(a+x)}$$

$$\int \epsilon^{\frac{1}{m(a+x)}+n}dx = \int \epsilon^z dx$$

$$z = \frac{1}{m(a+x)}+n \quad \therefore dz = -\frac{dx}{m(a+x)^2}$$

$$\therefore dx = -dz\left[m(a+x)^2\right]$$

$$(a+x)^2 = \frac{1}{m^2(z-n)^2}$$

$$\therefore dx = -dz\frac{1}{m(z-n)^2}$$

$$\therefore \int \epsilon^z dx = \int \frac{-\epsilon^z}{m(z-n)^2}dz$$

Let $(z-n) = v \quad \therefore dz = dv$

$$\therefore \int -\frac{\epsilon^z dz}{m(z-n)^2} = \frac{\epsilon^n}{m}\int -\frac{\epsilon^v}{v^2}dv =$$

$$\frac{\epsilon^n}{m}\left(\frac{\epsilon^v}{v} - \int \frac{\epsilon^v}{v}dv\right)$$

$$= \frac{\epsilon^z}{m(z-n)} - \frac{\epsilon^n}{m}\int \frac{\epsilon^{z-n}}{z-n}dz$$

$$y = a + \frac{\epsilon^m x}{am}$$

elementary weight of any portion δs ∝ $\frac{1}{(a+x)^2}$

$\therefore$ weight of AB ∝ $\int \frac{\sqrt{dx^2+dy^2}}{(a+x)^2}$

But $\frac{dy}{dx}$ ∝ as weight of AB $\therefore$

$$m\frac{dy}{dx} = \int_x \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{(a+x)^2}$$

$$\therefore m\,dp = \frac{\sqrt{1+p^2}}{(a+x)^2}dx$$

$$\frac{dx}{m(a+x)^2} = \frac{dp}{\sqrt{1+p^2}}$$

$$\therefore \int_x \frac{1}{m(a+x)^2} = \int_p \frac{1}{\sqrt{1+p^2}}$$

Let $a^2+2ax+x^2 = z$

$$2(a+x)dx = dz$$

$$\therefore dx = \frac{dz}{2\sqrt{z}}$$

$$\therefore \int_z \frac{1}{2m\sqrt{z^3}} = \frac{1}{2m}\int z^{-\frac{3}{2}}dz = -\frac{1}{m}z^{-\frac{1}{2}} = -\frac{1}{m\sqrt{z}}$$

$$\therefore -\frac{1+n(a+x)}{m(a+x)} = \log\left(\sqrt{1+p^2}+p\right)$$

$$\epsilon^{\frac{1}{m(a+x)}+n} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} + \frac{dy}{dx}$$

$$\epsilon^{\frac{2}{m(a+x)}+2n} - 2\epsilon^{\frac{1}{m(a+x)}+n}\frac{dy}{dx} = 1$$

$$\therefore dy = \frac{\epsilon^{\frac{2}{m(a+x)}+2n}-1}{2\epsilon^{\frac{1}{m(a+x)}+n}}dx$$

Int^d on preceding page

$\therefore$ height of AB $= a \tan\varphi$

height at C $= a \sec\varphi$

$\therefore$ height of AC $= \int \sqrt{dx^2 + dy^2}\, a \sec\varphi$

$= \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy\, a \sec\varphi =$

$= \int a \sec^2\varphi\, dy = a \tan\varphi$

$\therefore a \sec^2\varphi\, dy = a \sec^2\varphi\, d\varphi$

$\therefore y = \varphi + c$ [illegible]

~~$\int \sqrt{1 + \left(\frac{dx}{d\varphi}\right)^2}\, d\varphi\, a \sec\varphi = a \tan\varphi$~~

~~$\sqrt{1 + \left(\frac{dx}{d\varphi}\right)^2}$ [illegible]~~

~~$\sec\varphi = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$~~

$\frac{dx}{dy} = \frac{1}{\tan\varphi} \therefore \frac{dy}{dx} = \tan\varphi$

$\therefore x = \int_{\varphi} \tan\varphi = \int \frac{1}{\cos\varphi} + C$

$\therefore y = \varphi, \; x = \int \sec\varphi + C$ [illegible]

$\epsilon^{-x} \epsilon^{x} = \sec(y - C) \therefore y =$ [illegible]

$y = \epsilon^{x}$ Eqn to a log. Curve

$y = \epsilon^{x+c}$

$y = \epsilon = 2\varphi$

$\mu = \epsilon$ [illegible]

5. Proposition 8 proved differently.

Bisect the angle ABC by the line BD.

Then AD : DC :: 4 : 5 but

AD + DC = 6 ∴ AD = 2⅔ DC = 3⅓

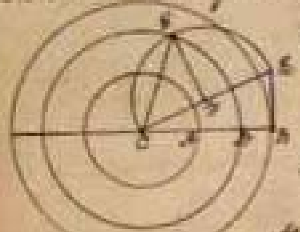
But 2⅔ : 4 :: 4 : 6 ∴ AD : AB :: AB : AC

∴ ABD, ABC are equiangular & ABD = $\frac{ABC}{2}$ = ACB.

2 : 8 : D.

(I.G. dux) (B.G. Tait)

6 To bisect the circular ring between the circumferences of two unequal concentric circles by the circumference of a circle also concentric with the former.



Investigation.

Let CD = r, CB = R, & CO = x

$\pi(R^2 - r^2)$ = area of ring

$\pi(x^2 - r^2)$ = half the area of ring

$\therefore 2x^2 = R^2 + r^2 \therefore x = \left(\frac{r^2 + R^2}{2}\right)^{\frac{1}{2}}$

Construction.

At B erect a perpendicular BE = CD, join CE which bisect in F and describe on it a semicircle. Raise a perpendicular at F cutting the semicircle in G, join CG, & with C for centre & radius CG describe a circle. It shall be that required.

(I.G. dux) (B.G. Tait)

7. If the diagonals of a parallelogram are equal it shall be a rectangle.

8.

Make OG = OC.

when ADEB = π, & AC = CB,

EAB = 30°.

BC · AB + BD · DA = DC²

(see next page)

Prove, BH² = EF · DH + FC · ED.

DE : EH :: HB : EA

∴ CH² = DE · EA

EF : AE :: ED : DH

∴ AE · ED = EF · DH

but

AC · ED = AE · ED + FC · ED.

$\therefore$ CH² = EF · DH + FC · ED. (2 : 8 : D.)

When weight ∝ ordinate

$y = a \sec\varphi$

$\int y\sqrt{dy^2 + dx^2} = a\tan\varphi$

$\therefore y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = a\sec^2\varphi$

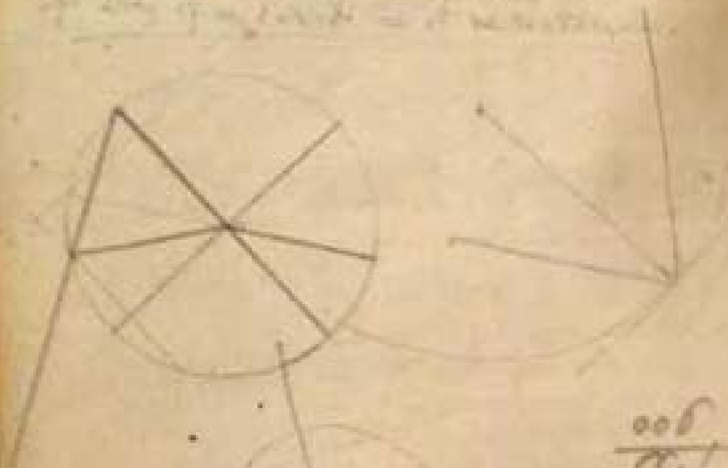
$\therefore \sec\varphi\, dy = \sec\varphi\, dx$

[illegible]

$= (x - x\theta)_{\frac{x}{2}} - \frac{\pi}{2}(x - x\theta)\frac{x}{2}$

[illegible]

[illegible]



1.

From A, to any [illegible] find a point in CB so that ADB = BOB.

Join AB, bisect it in [illegible] from which raise [illegible] perp. cutting CB in D. Join AD, DB which will be equal. About ADB describe a circle cutting [illegible] in O. Join AO, OB, AOD = DOB.

$\because$ AD = DB and they are chords of the same circle $\therefore$ AOD = DOB.

P.T. junr. (Which was to be done) P.G. Tait

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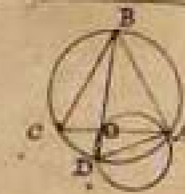
2. If in any triangle ABC, $AB^2 = BO^2 + AO \cdot OC$, ABC shall be isosceles.



About ABC describe a circle & produce BO to the circumference in D. Join AD and about AOD describe a circle.

$\because AB^2 = BO^2 + (AO \cdot OC$ that is$) BO \cdot OD = OB \cdot BD \therefore$ AB is a tangent to AOD $\therefore$ BAC = BDA = BCA.

P.T. junr (2 · S · B.) P.G. Tait

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3. If the sides of a triangle be 4, 5 & 6 respectively the angle BAE opposite 6 shall be double that ABE opposite 4.

Its is obvious that if a circle be described with centre B and radius 4 and we prove the part BE without the circle also = 4 the angles must be as stated.

From D draw GD perpendicular to AB and cutting it in G.

$(EG^2 - GA^2) = (ED^2 - AD^2) = 20 = (EG + GA)(EG - GA)$

& but $EG + GA = 5 \therefore EG - GA = 4 \therefore EG = \frac{9}{2}$ and

$GA = \frac{1}{2}$ but $AB = 2 AG = 1 \therefore BE = 4$.

From 3. [illegible] (2 · S · B) (P.G. Tait

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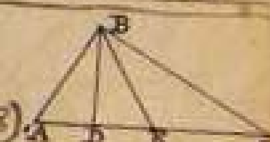
4. In any triangle ABC if a perpendicular be let fall from the vertex upon the base $BC^2 \sim AB^2 = DE \cdot AC$.



For $BC^2 \sim AB^2 = AD^2 \sim DC^2 =$

$(DC + AD)$ that is $AC) (AD \sim DC$ that is$) DE)$

From A.S. (2 · S · B) (P.G. Tait

Prop. If the [illegible] of the [illegible] hyp. [illegible] CD, [illegible] AB, [illegible] asymptotes [illegible] points E, D, C. AB || DE.

Area [illegible] $= \frac{CA}{2}(EB - DA)$

$= \frac{r}{2}(\varphi - \vartheta)(m) + \frac{\sin\vartheta \times \cos\vartheta}{2}$

$= \frac{r}{2}(m)(\vartheta) - \frac{\sin\vartheta \cos\vartheta}{2}$

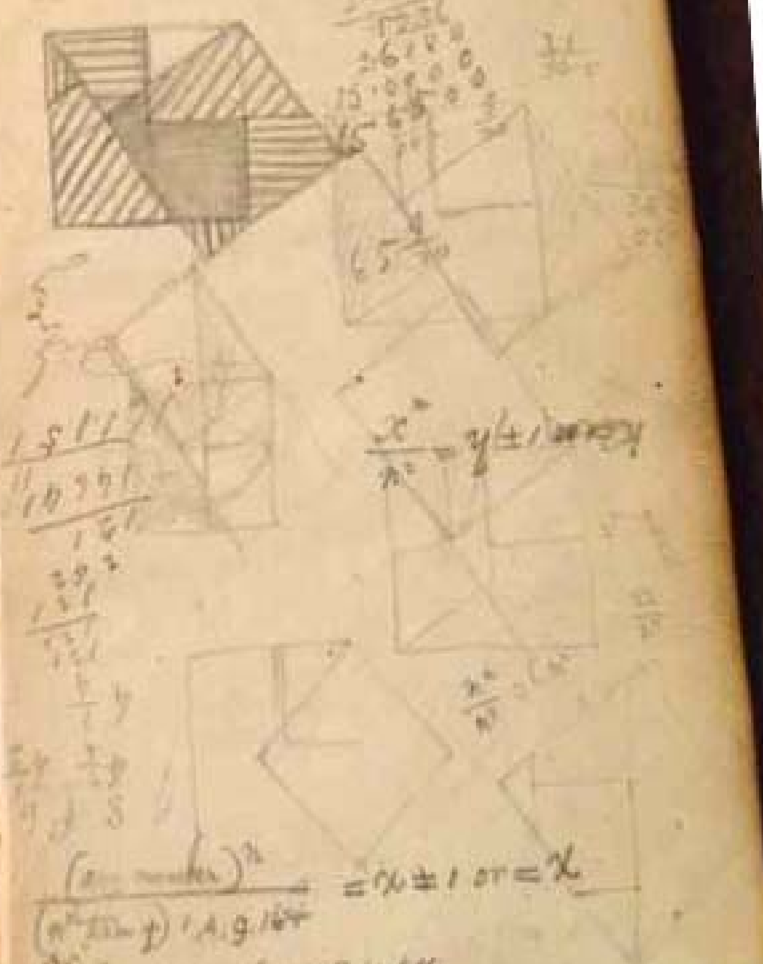
$\frac{r}{2}(m)(2\vartheta - 90) = \sin\vartheta \cdot \cos\vartheta$

$\frac{r}{2}(m)\left(\frac{\vartheta}{n} - \frac{1}{4}\right) = \sin\vartheta \cdot \cos\vartheta$

$\sqrt{r^2 - x^2} : x :: r : \left(\frac{rx}{\sqrt{r^2 - x^2}} - \varphi\right)\frac{r}{2}$

$= \frac{r}{4}\varphi - \frac{x}{2}\sqrt{r^2 - x^2}$

$\frac{m^2 x}{2} - \frac{r}{2}\sqrt{r^2 - x^2}\;\varphi = x\sqrt{r^2 - x^2}\;\varphi - $ [illegible]



$\frac{x^2}{n^2} = y \pm 1$ [illegible]

$\frac{(\ldots)^2}{(n^2 \ldots)}$ A. g. 164 $= x \pm 1$ or $= x$

Hyp. chord CD || AB.

Parab. 45° degrees – other [illegible] sections equated of 4th order.