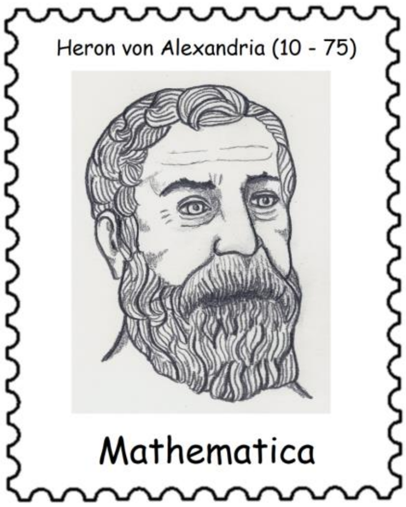
HERON OF ALEXANDRIA (10 – 75 AD)

by HEINZ KLAUS STRICK, Germany



(Drawing © Andreas Strick)

Whether HERON OF ALEXANDRIA looked as shown in the illustration is not reliably known, nor whether he actually lived in the period from 10 to 75 AD. Since HERON only referred to ARCHIMEDES (287 – 212 BC) in his writings, and his writings in turn were not cited before the time of the mathematician PAPPUS in the 4th century AD, his life dates were unclear for a long time.

It was not until 1938 that it could be proven that a report by HERON about a lunar eclipse referred to a celestial constellation that only existed in the year 62 AD. From this, the above-mentioned dates of his life were deduced.

HERON was a teacher and researcher at the *Museion*, the famous library and university in Alexandria. Some of his writings are elaborate textbooks, others are lecture manuscripts in style. The complete works of HERON represent something like an encyclopaedia of applied mathematics and engineering sciences of the Greco-Roman era; the writings contain many insights that date back to Egyptian and Babylonian times.

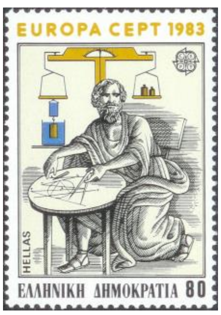
HERON's works were handed down over centuries, and it is not clear in all cases which parts were written by him and which were added to later. HERON's main work, *Metrica*, was long considered lost; only in 1896 was it rediscovered in the original Greek. Other works, such as the commentaries on EUCLID's *Elements* or the three-volume *Mechanica*, survived as translations by Islamic scholars.

Some historians of science expressed fundamental doubts about the originality of HERON's contributions. VAN DER WAERDEN, for example, wrote: *Let us be glad that we have the masterpieces of ARCHIMEDES and APOLLONIUS, and let us not mourn the numerous lost little books of arithmetic of HERON's kind.*

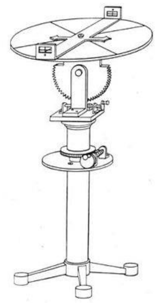
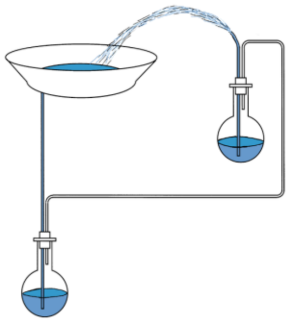
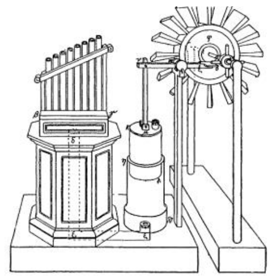
Without a doubt, however, HERON was the last important textbook author of antiquity who expounded the practical use of mathematical methods. In this sense, his works are almost comparable to the more theoretical elements of EUCLID in their importance for the following centuries.

HERON's book *Pneumatica* contains over 100 detailed descriptions of devices that served solely to entertain people: Devices for the independent opening of temple doors, holy water dispensers, water organs and much more. The Greek stamp on the right shows a replica of an automatic theatre in which various scenes with moving figures were shown one after the other with the help of steam and water pressure, cogwheels and rope connections.

The precursor of the steam engine, the *Aeolipile*, and the seemingly endless *HERON fountain* also served to amaze the population and had no practical purpose.



In contrast, the book *Mechanica* summarises the knowledge (largely due to ARCHIMEDES) of simple machines such as levers, wedges, pulleys and screws, but also contains explanations of cranes and wine presses.

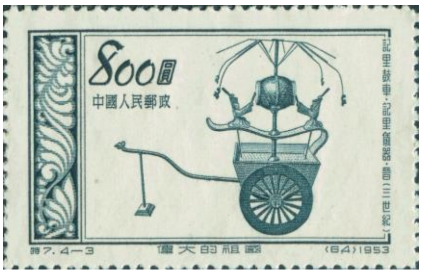
			
HERON's Dioptra	HERON's Aeropile	HERON's Fountain	HERON's Wind Organ

Graphics from Wikipedia

HERON's book *Dioptra* deals with various methods of surveying; the word *dioptra* stands for a measuring device that corresponds to our modern theodolite: a dioptra consists of a frame with a horizontally rotating crossbar, at the end of which sighting devices are attached and with which right angles can also be set. The horizontal alignment of the sighting device is carried out at the four ends of the crossbar according to the principle of communicating tubes.

The book explains all the methods that are part of a surveyor's daily routine – many of which have been handed down from the Egyptian rope-stretchers (*harpedonapts*) – such as levelling a terrain, staking out a straight line between two points that are not visible at the same time, determining the distance of inaccessible points, reconstructing rectangular fields of which only individual boundary stones have survived.

Also presented in *Dioptra* is a *hodometer* (ὁδός = path; μέτρο = measure), a wagon with which an astonishingly accurate measurement of a distance could be made: The revolutions of the carriage wheels are gradually translated into slower movements with the help of cogwheels. Although HERON is considered the inventor of the hodometer, references to this method of measurement can be found 100 years earlier in the work of the Roman architect and engineer VITRUVIUS, whose diagram of the proportions of the human body was made famous by LEONARDO DA VINCI.
A similar measuring carriage was used in China at the same time; the mathematician and astronomer ZHANG HENG (78 – 139) is considered the inventor here.

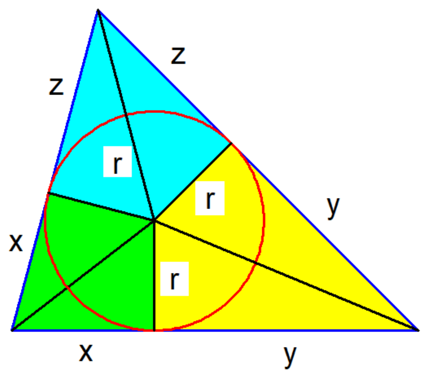


In a chapter on astronomical measurement methods, HERON explains how to determine the distance between two places, e.g. the cities of Rome and Alexandria:

The occurrence and the end of a lunar eclipse is observed in both places at the same time; from the measurement data (true local time and celestial coordinates), the geographical latitude and longitude difference of the two places can be determined and thus the distance calculated. This method was not mentioned in the writings of PTOLEMY (85 – 165 AD); therefore, for a long time it was erroneously assumed that HERON lived *after* PTOLEMY.

Finally, the *Dioptra* contains a formula for calculating the area *A* of a triangle from the side lengths *a, b, c*, i.e. with semiperimeter (half the circumference) $s=\frac{1}{2}\cdot(a+b+c)$, which is called HERON's triangle formula:

$$A_\Delta=\frac{1}{4}\cdot\sqrt{(a+b+c)\cdot(a+b-c)\cdot(a-b+c)\cdot(-a+b+c)}=\sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-c)}$$

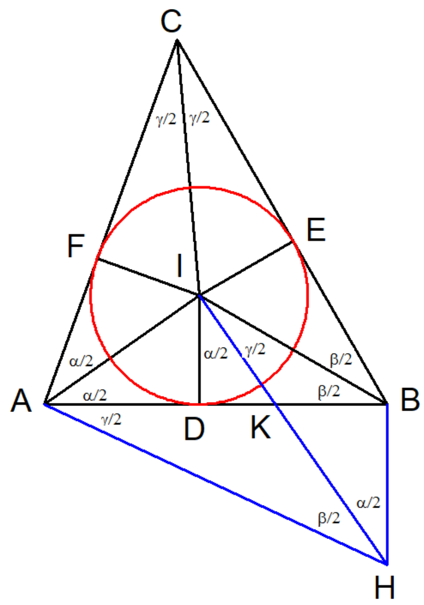


The figure on the left shows that a triangle is divided into three pairs of congruent right triangles by the bisectors and the perpendiculars from the centre of the incircle to the sides of the triangle (= incircle radius *r*). The area *A* of a triangle can therefore be calculated with the help of the incircle radius *r* and the half perimeter *s*:

$A_\Delta=2\cdot(\frac{1}{2}\cdot x\cdot r+\frac{1}{2}\cdot y\cdot r+\frac{1}{2}\cdot z\cdot r)$, thus

$A_\Delta=(x+y+z)\cdot r=s\cdot r$

HERON proves the triangle formula with the help of the second figure:



The point H is constructed as the intersection of the perpendicular to AB through B and the perpendicular to AI through I. H is connected to A.

Because of the two right angles at I and B, a THALES circle could be drawn over the line AH as the diameter.

Therefore $AHBI$ is a cyclic quadrilateral in which opposing angles have an angular sum of 180°. This gives the angles shown in the figure.

In the triangles that are similar to each other, the result is then:

From the similarity $AHB \sim CIE$ follows:

$BA : HB = (x + y) : HB = EC : IE = z : r$, i.e. $(x + y) : z = HB : r$.

From the similarity $BKH \sim DKI$ follows: $HB : BK = ID : DK = r : DK$,

i.e. $HB : r = BK : DK$, in summary: $(x + y) : z = HB : r = BK : DK$.

A tricky transformation follows:

$$\frac{x+y}{z}=\frac{BK}{DK}\Leftrightarrow\frac{x+y}{z}+1=\frac{BK}{DK}+1\Leftrightarrow\frac{x+y+z}{z}=\frac{BK+DK}{DK}\Leftrightarrow\frac{s}{z}=\frac{y}{DK}\Leftrightarrow s\cdot DK=y\cdot z$$

Since the triangle AKI is right-angled (by construction), according to the right triangle altitude theorem:

$AD - DK = DI^2$, thus $x - DK = r^2$. From this then follows

$$A_\Delta^{\,2}=s^2\cdot r^2=s^2\cdot x\cdot DK=s\cdot x\cdot(s\cdot DK)=s\cdot x\cdot(y\cdot z)=s\cdot(s-(y+z))\cdot(s-(x+z))\cdot(s-(x+y))$$

and therefore $A_\Delta^{\,2}=s\cdot(s-a)\cdot(s-b)\cdot(s-c)$, thus $A_\Delta=\sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-c)}$.

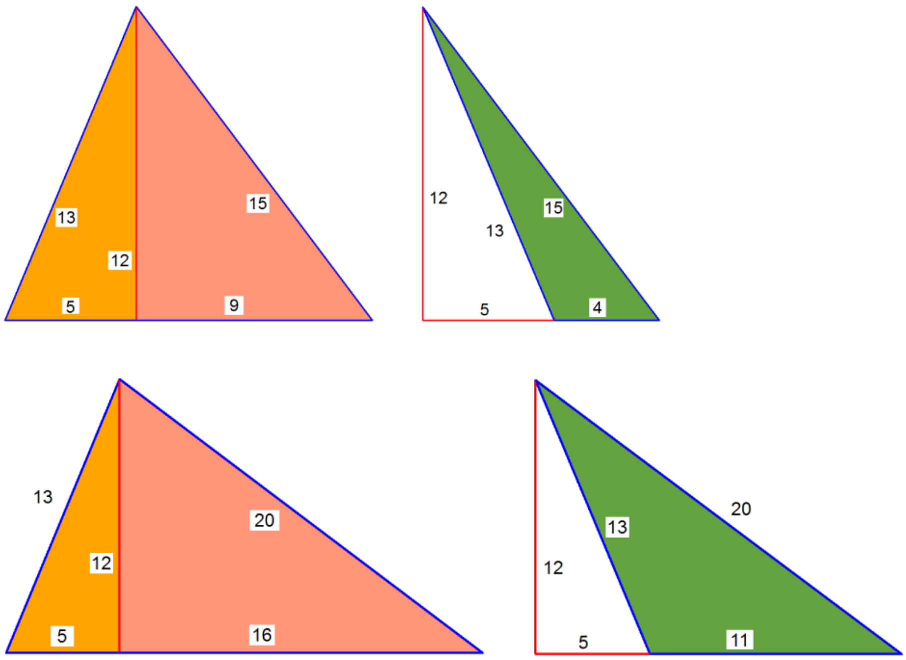
From HERON's formula it follows that the area measure of a triangle is generally an irrational number, even if the side lengths are rational. On the other hand, it follows from the area formula $A_{\Delta} = \frac{1}{2} \cdot g \cdot h$:

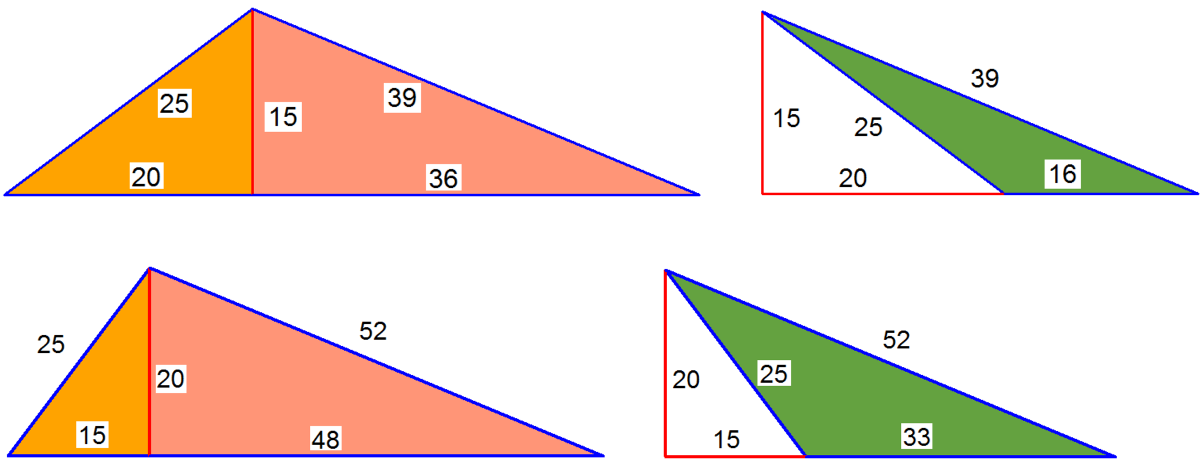
- *If the side lengths and one altitude are rational, then the other altitudes and the area are also rational.*

Triangles with rational sides and heights are called *HERON triangles*.

These triangles are made up of two right-angled triangles whose side lengths are rational PYTHGOREan triples of numbers.

(Graphics from *Mathematics is beautiful*, Springer, 2021)





Since one determines the area of a triangle given by the three sides – according to HERON's formula – by taking the square root, HERON gives a simple method for this. It still bears HERON's name today, although HERON only describes a method that was already known to the Babylonians 2000 years earlier and is therefore also called *Babylonian root extraction*.

HERON explains the method in the 1st volume of his *Metrica* using the example of the result $A^2 = 720$:

Since 729 is the nearest square number, divide 720 *by* 27. *The result is* $26\frac{2}{3}$. *Add 27 and this gives* $53\frac{2}{3}$ *and half of this is* $26\frac{5}{6}$. *Multiplying* $26\frac{5}{6}$ *by itself gives* $720\frac{1}{36}$, *which is only* $\frac{1}{36}$ *different from* 720. *If you want an even smaller difference, start with* $26\frac{5}{6}$ *instead of* 27.

The 1st volume of the *Metrica* contains instructions on how to calculate the area of polygons and the surface area of a cone, a cylinder, a prism, a pyramid and a sphere.

In the 2nd volume, the volumes of these solids are calculated.

The 3rd volume deals with the problem of how to divide given surfaces or bodies in a given ratio, where HERON takes over much of EUCLID.

For calculating the volume of a truncated pyramid (correspondingly also of a truncated cone) the formula applies: $V = \frac{1}{3} \cdot h \cdot \left(A_1 + A_2 + \sqrt{A_1 \cdot A_2}\right)$, where A_1 or A_2 is the area of the base or top surface of the frustum and h is the height. One can also write this down in the following form: $V = h \cdot \left(\frac{2}{3} \cdot \frac{1}{2} \cdot (A_1 + A_2) + \frac{1}{3} \cdot \sqrt{A_1 \cdot A_2}\right)$, i.e. as a weighted mean of the arithmetic and geometric mean of the area sizes. The term $\frac{1}{3} \cdot \left(x + y + \sqrt{x \cdot y}\right)$ is therefore sometimes referred to as *HERON's mean* of the quantities x and y.

First published 2013 by Spektrum der Wissenschaft Verlagsgesellschaft Heidelberg

https://www.spektrum.de/wissen/heron-von-alexandria-10-75/1198440

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