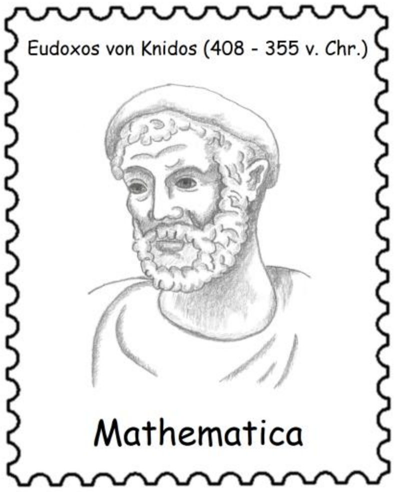
EUDOXUS OF KNIDOS (408 – 355 BC)

by HEINZ KLAUS STRICK, Germany



Even though we do not even know the exact titles of his mathematical works and only fragments of his other writings have survived, we can say that EUDOXUS OF KNIDOS was one of the most important mathematicians of antiquity.

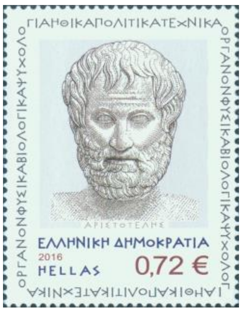
It is known that the scientist, who was born in Knidos (Asia Minor), travelled to Taranto (a Greek colony in southern Italy) to pursue his first mathematical studies with ARCHYTAS, one of the successors of PYTHAGORAS.

(drawing © Andreas Strick)

In Sicily, he acquired medical knowledge from PHILISTON, in Athens he probably attended the lectures of PLATO and other philosophers of the *Academy*, and in Heliopolis (Egypt) he was introduced to the techniques of astronomical observation by the priests.

He then founded his own school in Kyzikos, a Greek colony on the southern coast of the Sea of Marmara, and gathered numerous students around him.

Around 368, he visited Athens a second time, accompanied by his students, and then returned to his native city of Knidos as a respected citizen, where he built an observatory. His astronomical observations formed the basis for (at least) one work which HIPPARCHUS OF RHODES (190 - 120 BC) used for his investigations and reflections, as the latter gratefully reported.

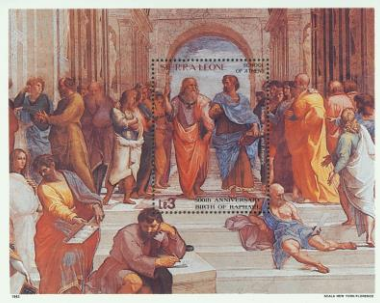


ARISTOTLE (384 - 322 B.C.) reported that EUDOXUS developed a system to describe the movements of the planets. This consisted of 27 spheres, in the centre of which was the Earth.

EUDOXUS also wrote a work on geography consisting of seven volumes, in which he described the countries and peoples of the known world, explained the political systems in these countries and reported on the religious ideas of the peoples. This work is also lost, but was cited by numerous authors of antiquity who lived later.

The discovery by the Pythagorean HIPPASOS OF METAPONT that not all quantities occurring in geometry are commensurable, i.e. measurable with a common measure, had shaken the doctrine that "everything is number" that had been accepted until around the year 500 BC.

For example, the ratio of the length of a diagonal of a square to the side length of the square cannot be described by the ratio of two natural numbers.



EUDOXUS found an ingenious way of dealing with this problem. EUCLID later (around the year 300 BC) adopted EUDOXUS's theory of proportions as Book V of the *Elements*.

First, EUDOXUS defined what is meant by a ratio:

- *A ratio is the relationship of two comparable things in size* (V.3).
- *A ratio indicates how often the first size exceeds the second when multiplied by the second* (V.4).

Then comes the – at first glance – seemingly complicated but extremely clever definition V.5:

- *Sizes are in the same ratio, the first to the second as the third to the fourth, if for any but equal multiples of the first and the third size and for any but equal multiples of the second and fourth size it holds that the multiples considered in pairs are either both greater or both equal or both smaller.*

Expressed in today's usual notation:

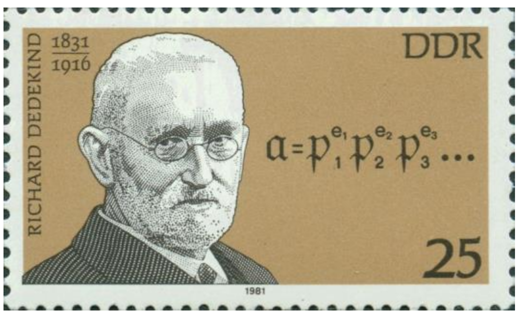
- Two ratios $a:b$ and $c:d$ of quantities a, b, c, d coincide exactly, i.e. $a:b=c:d$

  if for any multiples $m, n \in \mathbb{N}$ we have:

  From $m \cdot a > n \cdot b$ it follows that $m \cdot c > n \cdot d$; from $m \cdot a = n \cdot b$ it follows that $m \cdot c = n \cdot d$; from $m \cdot a < n \cdot b$ it follows that $m \cdot c < n \cdot d$.

The genius of the Eudoxian approach is that its definition is applicable to both rational and irrational quantities: for rational quantities, the case of equality occurs, that is, multiples m, n can be specified for which equality applies.

But if the quantities a and b are not commensurable, then there are both rational numbers $\frac{m}{n}$ for which $\frac{m}{n} > \frac{b}{a}$ as well as some for which $\frac{m}{n} < \frac{b}{a}$ holds.

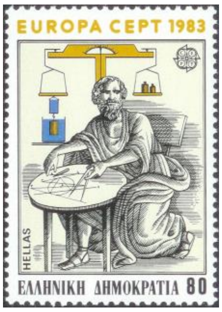


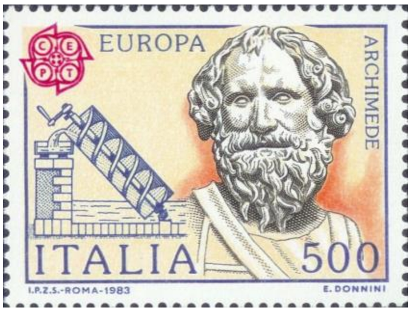
In principle, this was nothing other than the idea that a number splits the set of real numbers into two disjoint subsets.

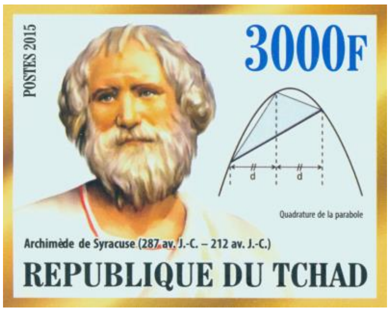
But it took more than 2200 years before RICHARD DEDEKIND implemented this idea by means of the so-called *DEDEKIND cuts*.

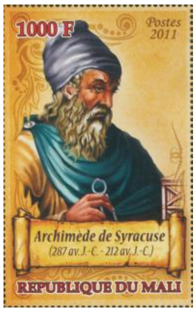
At the beginning of Book X of the *Elements* of EUCLID, one finds a method of calculating area that has been called the *exhaustion method* since the 17th century:

- *If two unequal sizes are given and one takes away more than half of the larger one, of the remainder again more than half and so on, then one eventually arrives at a remainder that is smaller than the given smaller size*.









With the help of this exhaustion method, the dimension of an area can be determined as precisely as desired, for example that of a circle by inscribing polygons. The theorem is based on an application of the so-called *Archimedes' Axiom*, which states that for every two sizes, a multiple of one size can be formed so that it is larger than the other size. It would have been quite appropriate if this principle had been named after Eudoxus; for he was expressly described by Archimedes as the originator of the axiom.

Book XII of the *Elements* dealt with surface areas and volumes. These explanations were also predominantly based on theorems and proofs that Euclid took over from Eudoxus.

The proof of Theorem 2:

- *The areas of circles behave like the squares of their diameters.*

was carried out using the method of indirect proof (*reductio ad absurdum*). The assumption that the ratio of the areas of the circles was smaller than the ratio of the squares of the diameters led to a contradiction, as did the assumption that the ratio was larger.

The proof of proposition 18 was analogous:

- *Volumes of spheres behave like cubes of their diameters*.

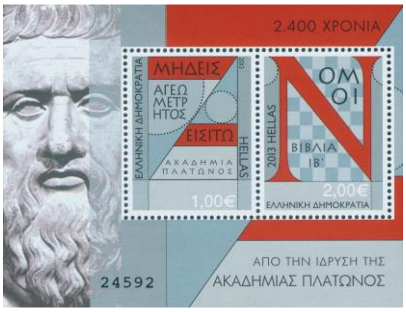
The propositions between proposition 2 and proposition 18 dealt with the calculation of the volume of a pyramid or a cone.

Democritus (460 - 370 B.C.) already knew the formulae, but as Archimedes explained in his *On the Sphere and the Cylinder*, the proof of the formulae was first given by Eudoxus.

First he explained how pyramids with a triangular base could be divided into two equal pyramids similar to the whole pyramid and two prisms. Then he showed that the volumes of pyramids of the same height with triangular (or generally polygonal) bases behaved like the areas of the bases. In the next step, he showed how one could decompose a prism into three pyramids of equal volume with triangular bases. From the theorem that the volumes of pyramids similar to each other behaved like the cubes of corresponding edge lengths and the theorem that the bases of pyramids of equal volume were inversely proportional to the heights, it finally followed that the volume of a pyramid was exactly one third of the volume of a prism with the same base and the same height.

Eudoxus also dealt with the Delian problem of cube doubling. Eratosthenes (276 - 194 BC) reported that Eudoxus, "the god-like", found a graphical solution to the problem. Unfortunately, no further details have survived.

Plato, however, is said to have criticised the procedure because it would "contaminate mathematics".



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https://www.spektrum.de/wissen/eudoxos-von-knidos-der-schoepfer-der-exhaustionsmethode/1331747

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