An early manuscript of MacLaurin's: mathematical modelling of the forces of good; some remarks on fluids.

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1. Introduction

Colin MacLaurin (1698–1746) graduated Master of Arts from the University of Glasgow in 1713 and became Professor of Mathematics at Marischal College, Aberdeen in 1717. Little is known about his activities between these events; according to Patrick Murdoch, editor of [10] and author of the biographical sketch of MacLaurin contained therein, he spent a further year at Glasgow in the study of Divinity and then withdrew to the home of his uncle, the Rev. Daniel MacLaurin, minister of Kilfinan parish, Argyllshire, where he continued his studies in mathematics and philosophy and read classic authors. In fact, Daniel MacLaurin, to whom Colin affectionately dedicated his MA dissertation, (1) had been much involved in his nephew's upbringing: Colin's father, the Rev. John MacLaurin, minister of Kilmodan parish, Argyllshire, died some six weeks after the boy's birth and his mother died in 1707.

Daniel MacLaurin sent his nephew's dissertation to his colleague the Rev. Colin Campbell, minister of Ardchattan and Muckairn parish, Argyllshire. Campbell, who enjoyed a considerable reputation in natural philosophy and mathematics, responded enthusiastically about the work and included his own commentary on it. (2) All this was conveyed to the young mathematician, who now wrote to Campbell (Sept. 12, 1714):

> '... This satisfaction received no small addition from the sight of a letter my Uncle had received from you concerning my Theses not only on the score of its affording an uncontestable proof that fame had not overrated your skill, but also because of the characters it bore of your civility & discretion in giving so favourable treatment to that discourse which I am sensible was lyable to many

(1) The dissertation is entitled *De Gravitate, aliisque Viribus Naturalis* (Concerning Gravity and other Natural Forces). I have reproduced it along with a translation and commentary in Part I of [17].

(2) Concerning Campbell see J. Henry's article in [7]. Campbell conducted an extensive correspondence with distinguished figures from mathematics, natural philosophy and other areas of science. See: 'Letters from mathematicians to Colin Campbell, and mathematical papers' in the Colin Campbell Collection, Edinburgh University Library (GB 237 Coll–38). Concerning the cited letters: Daniel MacLaurin's letter to Campbell is MS 3099.15.4; there is a typed copy of Campbell's reply in MS 3099.15; Colin MacLaurin's letter to Campbell is MS 3099.15.1 and the enclosure, with which this article is concerned, is MS 3099.15.6 – there is a copy of the letter in the National Library of Scotland (MS 3440.f32) and it is reproduced as Letter 116 in [12]. Typed copies of all the MacLaurin items, including the enclosure, will be found at Glasgow University Library among papers collected by John Eaton (see especially MS Gen. 1332).

exceptions & very much deserved your censure. ...'

After some comments about his dissertation, the universality of the law of gravitation and theological matters, he added:

> 'But since I have mentioned the use of mathematicks I shal beg your pardon for troubling you with some thoughts I have relating to their use in morality. I have sent them to you under the title of De Viribus Mentium Bonipetis.'

The letter contained an enclosure of seventeen pages, (3) the first ten of which form *De Viribus Mentium Bonipetis* (Concerning the good-seeking forces of minds). In this article MacLaurin is concerned with 'good things' (bona), but these are not formally defined. He first discusses their intensities – essentially time-dependent density functions – and then presents several mathematical models for the forces by which our minds are attracted to 'good things' over time. Ideas of exponential decay are encountered; of course, he does not use this terminology or associated notation, which belong to a later time. There are also some interesting applications of integration and a few comments on moral or theological matters.

An outline of *De Viribus Mentium Bonipetis* along with some background to it has been given by J. V. Grabiner in [5] (see also [6]); she also cites the unpublished Ph.D thesis of E. L. Sageng [15], which has rather more discussion of this item. MacLaurin was to become one of the greatest exponents in his time of Newtonian ideas, especially through his *Treatise of Fluxions* (1742) [9], (4) and Grabiner sees his article as a 'sort of trial run of Maclaurin's use of the Newtonian style' [5]. It is no doubt with justification that she refers in [6] to the work as 'a Latin essay still (perhaps mercifully) unpublished today'. Nevertheless, I find some interesting mathematics in it and in the rest of the enclosure and I am sure that MacLaurin would not have submitted to Campbell's scrutiny anything that he did not consider to be a worthwhile development. On the other hand Campbell did not reply to MacLaurin's letter (5) and I do not know of any further work of this type by MacLaurin, although he did write more on theological matters (see Book IV, Chapter IX of [10] or [11]). Perhaps *De Viribus Mentium Bonipetis* has to be seen as a one-off, youthful exercise composed at a time when MacLaurin was seeking to apply his mathematics to his other great interest, theology. As pointed out by Grabiner [6] others were applying mathematics to theological matters about this time; indeed, MacLaurin

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(3) The first eleven pages are numbered and p. 11 is headed 'Prop. altera'. These annotations appear to have been added later, possibly in a different hand.

(4) Grabiner has discussed the significance of this work in [4].

(5) See MacLaurin's second letter to Campbell (July 6, 1720; MS 3099.15.2, [12], Letter 117).

may have been influenced by the earlier work *Theologiae Christianae Principia Mathematica* (1699) by fellow Scot John Craig (or Craigie) (1663–1731) [1] (see also [2], which contains a translation of part of [1]). This controversial work, which uses probabilistic ideas, has been seen as a parody of Newton's *Principia* but has also received some more sympathetic comment. (6) Sageng (along with Grabiner) notes Francis Hutcheson's *An Inquiry into the Original of our Ideas of Beauty and Virtue* (1725) [8] which includes several simple mathematical formulae in an article headed: To find a *universal Canon* to compute the *Morality* of any Actions, ... (7)

MacLaurin concludes his letter to Campbell by telling him that he has been applying fluxional methods to prove many of Newton's results from the *Principia* and as an example of this he is also sending

> 'another paper being some easy fluxional demonstrations of the Prop 21, 22 of the 2d Book of Neuton's principles with what he has asserted without demonstration in their Scholiums together with some other smal Improvements.'

This material constitutes the rest of the enclosure and is concerned with the effect of centripetal force on the density and compression of a fluid; a similar range of mathematical techniques is applied to determine density under various assumptions concerning the other quantities. Both [9] and [10] contain material on fluids, notably in connection with the figure of the earth and the tides (cf. Part III of [17]), but I have not found anything in them which is directly related to the present material.

Section 2 of this article contains my translation of *De Viribus Mentium Bonipetis*. I

(6) See A. I. Dale's article on Craig in [7]. Craig's work is noted by Todhunter in [16], but largely to record negative views of it by others. It is perhaps of interest to note that there are seven letters from Craig in the Colin Campbell Collection (MS 3099.8; see footnote (2)). MacLaurin met Craig in London in 1719 (see letter of July 6, 1720 cited in footnote (5)).

(7) See [8], Treatise II: Section III, Articles XI, XII, XV. The following assertion from Article XV seems worth recording here: 'The applying of a *mathematical Calculation* to *moral Subjects*, may appear perhaps at first *extravagant* and *wild*; but some Corollarys, which are easily and certainly deduc'd below *, may shew the Conveniency of this attempt, if it could be further pursu'd.' (The ∗ indicates a footnote directing the reader to Sect. 7, Art. VI, VII.)

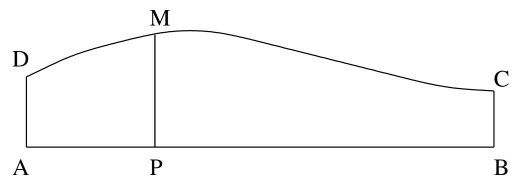
Francis Hutcheson (1694–1746), an Ulsterman, overlapped with MacLaurin at Glasgow University. He became Professor of Moral Philosophy there in 1730. See his entry in [7] by J. Moore for further details. MacLaurin wrote to Hutcheson in 1728 recalling their earlier association and praising [8] ([12], Letter 13).

have kept it quite literal and generally I have not attempted to paraphrase MacLaurin's text. The translation is accompanied in Section 3 by some notes which are intended to clarify MacLaurin's mathematical manipulations and assertions. The roman numeral superscripts which appear within my translation refer the reader to the corresponding notes; my footnotes are indicated throughout by arabic numerals. Section 4 contains a translation of the material on fluids along with my commentary and some relevant extracts from Newton. Since the original manuscript is not always easily legible I have attempted to reproduce (with some editorial licence) in Section 5 the Latin text of the whole enclosure. I hope that my efforts may be of some use to scholars with interests in MacLaurin or the Colin Campbell Collection.

2. Translation of De Viribus Mentium Bonipetis

Concerning the good-seeking forces of minds

Our minds, trying to reach every small part of apparent goodness in any good thing, are carried necessarily to the good thing itself by a force which is the sum of the forces by which its individual particles are sought. But if equal parts are sought with equal forces, or if the good-seeking accelerating force of all is the same, the sum of the forces with which the parts of any good thing are sought will be as the sum of the parts or as the quantity of goodness in that good thing: and consequently, the forces with which our minds are carried into different good things are (other things being equal) as the quantities of goodness in those good things. Therefore, in order that we may understand the schemes of those forces, the quantities of those good things require to be elucidated first of all.



Therefore, let the straight line AB represent the duration of the good thing, at its individual points P let normals PM be erected, which are as the intensities of the good thing (i.e. the instantaneous goodnesses) at the end of the time AP, and, if AD is the intensity with which the good thing begins, and BC that with which it ceases to exist, then the total good thing of duration AB will be as the area ADCB. For the total good thing is the sum of the instantaneous goodnesses, or as the sum of the ordinates PM, and so as the curvilinear area ADMCB or if the intensity PM is called I and the good thing which has passed by the time AP is called B, and the time AP t, there will result from the definition of intensity: [8]

$$\frac{\dot{B}}{\dot{t}} = I\,, \quad \dot{B} = I\dot{t} \quad \& \quad B = \mathcal{F}\, I\dot{t} = \mathrm{ADMP}\,,$$

therefore the good thing of duration AB = ADCB. If the apparent intensities and durations are substituted for the real intensities and durations, the apparent quantity of the good thing can be found by means of this proposition.

Note: these ideas and also others which follow can easily be applied to bad things.

By means of this proposition the maximum or minimum intensity of any good or bad thing is found by the common methods of maxima and minima, for if PM is the

(8) In the following expressions the symbol $\mathcal{F}$ stands for *fluent*, essentially the integral. MacLaurin notes this at a later occurrence in the second part of his manuscript (see Section 4).

maximum or minimum ordinate, the intensity is of the same order. But if the good thing is not continuous, but some distance of time occurs between its various parts, it can be regarded as the aggregate of the various lesser good things whose individual parts, determined by this proposition and added together, make it up.

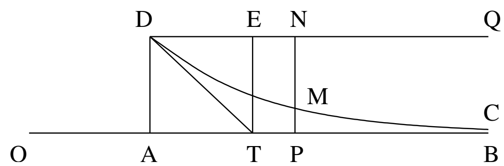
Moreover, it is clear from this that very many good things whose durations are infinite, are themselves finite; for there exist very many curvilinear areas extended to infinity or infinitely long, which neverthless are finite. The area between the cissoid and its asymptote [(i)] and the area between hyperbolas of any power (the conic alone excepted) and their asymptotes [(ii)] are of this type. But more particularly if the intensities of any good thing decrease in a geometric progression while the times increase in an arithmetic progression, then in this case the curve DMC will be the common logarithm [(iii)] (see the figure on page 8) [(9)] and the infinite area ADCB = ATED, where it is supposed that DT touches the curve at D.[(iv)] Therefore a good thing of this type of infinite duration AB is equal to a uniform good thing of duration AT and of constant intensity AD. In the same way, if any good thing observed on one day has a given quantity but on the following day only half of it and decreases thus indefinitely so that the good thing observed one day is half of that of the preceding day and twice that of the next day: I say that the total good thing of this type continued for infinitely many days is only twice the good thing which was observed on the first day, which is clear from the fact that

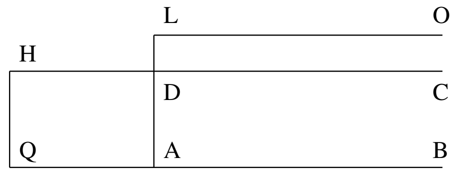
$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\ \&\text{c}:=2\,.$$

If the intensities of the good thing increase uniformly taking their beginning from nothing, the intensities will be directly proportional to the durations, and the good things themselves will be as the squares of the durations. But if the intensity is as some positive power of the duration passed through, the good things will nevertheless be perpetually in a ratio composed of the durations and the intensities; and consequently, if the durations of them are infinite, the good things themselves will be infinitely infinite.[(v)]

Also uniform good things, whose intensities are invariable, are as the durations and intensities jointly, since the figure ADCB now becomes a parallelogram.

[(9)] The reference is to p. 8 of MacLaurin's manuscript. Here the appropriate diagram is on p. 9, but is reproduced below for convenience.





1°. Hence a good thing of this type whose duration is infinite will itself be infinite. Hence:

2°. It is infinitely more preferable to increase the intensity of an infinitely lasting good thing of this type by the smallest possible quantity than to extend its duration as far as possible; for by adding DL to the intensity AD of the infinite good thing ADCB, we adjoin the infinite increase DLOC to it; but by adding AQ to the infinite duration AB the increment of the finite good thing AQHD results.

Therefore the complaints of good men about the miseries of this life must be removed by the consideration of the increment which is added as a result to the intensity of future eternal happiness, by which it comes about (by chance) that their total happiness altogether is greater than if it had started from the beginning of their existence and man had never fallen.[10] Moreover it seems to be quite clear that the intensity is increased on account of the fact that gratitude is greater and very many virtues are exercised for which there would be scarcely any place if all innocent men were to conduct their life without any bad things or misfortunes.

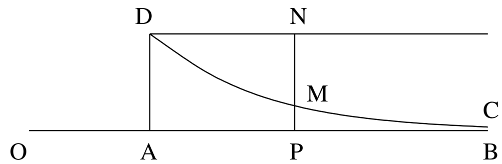
3°. In order that two good things of this type be equal, their intensities must be reciprocally proportional to their durations.[vi]

These things have been introduced concerning the measurement of good things in general; something more particular will have to be said about them later.

Hitherto, we have supposed the good-seeking accelerating force,[vii] or the force by which a given part of the good thing is sought, to be constant and everywhere the same. However, experience teaches that it is varied according to the different distance of time and from other causes. Let it be supposed that the good thing is uniform. If in the

[10] In translating this lengthy and complicated sentence I have recast the first part in passive form since I could not find an easy way of incorporating the original active form (the consideration … must remove the complaints …). Sageng also gives a translation of this sentence and the next in [15]. While his is similar, it does conclude with '…if all men lived their lives as innocents without any evils or misfortunes.' Perhaps this makes more sense, but I have retained my version, mindful of Latin's love of what one of my teachers used to call a 'noun sandwich': where a noun is modified by two adjectives the order is adjective–noun–adjective; in the present case we have 'omnes homines innocentes' (see p. 31, line 6).

given PN the PM are taken proportional to the good-seeking accelerating forces at the distances OP, the area ADMP will be as the force by which the mind is driven towards the good thing APND.



First of all, therefore, let the good-seeking force be supposed to be reciprocally proportional to the distance of the time,[(viii)] and the curve DMC will be the hyperbola of Apollonius drawn with centre O and asymptotes OD, OB. [(11)] Under this hypothesis:

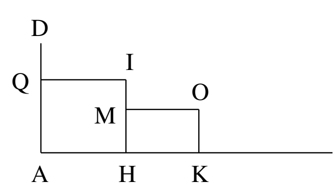
1°. If the durations of the good thing, increased by the distance OA, are taken in geometric progression, the forces with which the mind tends to those good things will be in arithmetic progression and the accelerating forces will be in geometric progression.

2°. A good thing of infinite duration attracts the mind to it with an infinite force, for the hyperbolic area ADCB is infinite.

3°. According to this hypothesis good things which are as the distances of the times of the mind from them, are equally sought: as where the good things which are as the numbers 1, 2, 3, 4, 5, 6 at distances of the times which are respectively as 1, 2, 3, 4, 5, 6 will be equally sought. For example, any good thing separated from the present by one hour and its double to be taken after two hours would be sought equally according to this hypothesis.

4°. The force into the present good thing is infinitely greater than the force into the same good thing removed even by a very small distance. And never at any non-infinite distance is it nothing.

And these two last Corollaries trouble me to the extent that I am of the opinion that this hypothesis is not true.



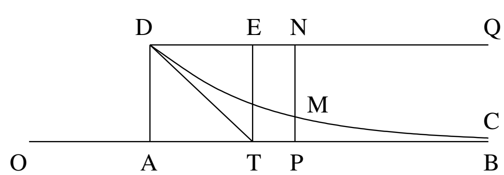
Let our mind be driven towards any present good thing by a force as AD; furthermore at distance AH, as HI. Now let it be supposed that it seeks the present good thing with a force as AQ (= HI) and at distance AH it seeks it with a force which is to AQ

[(11)] In MacLaurin's diagram OD is certainly not an asymptote. Perhaps he means the limiting position of OD as A $\rightarrow$ O.

as HI : AD, or as HM. But if HK is taken equal to AH on account of the same ratios as AQ (= HI) was reduced into MH, now HI will become OK (= MH); thus AD, HI, KO are continued proportionals; and so on where it is probable that the good-seeking accelerating forces decrease in a geometric progression if the times increase in an arithmetic progression.[ix]

According to this hypothesis the decreases or fluxions of the forces will be as the forces themselves (this is shown by Lemma 1 of Book II of Newton's *Principia*).[12] This hypothesis is quite in accordance with the malice and diligence of the Tempter, who probably applies greater effort to the containment of force the greater it is; hence it is very probable that Satan tries to establish this law in our minds or at least to suggest it while we are considering the good things of the future life.

This hypothesis is confirmed somewhat by experience: for the excess of the force into a good thing at distance one hour from this instant over the force into the good thing to be obtained after two hours, is greater than the excess of force into the good thing removed by a year over the force into the good thing to be obtained after a year and one hour. We are not so troubled about this difference as about the former whence it is clear that the good-seeking accelerating force does not decrease uniformly, so that any particle of time takes away the same quantity from it. But rather it decreases more rapidly while it is greater and more slowly when it becomes smaller; as when the decrease of force is as the force itself or some power of it.



This having been put in place:[13]

1°. Since the area ADCB = ADET the force into the infinite good thing ADEQBA will be equal to the force into the good thing ATED and the force AD will be continued uniform over the time AT. Therefore the force with which the infinite good thing is sought according to this law, is finite and it can be equalled or even overcome by temporary and finite good things. And hence comes forth a means for explaining a

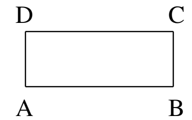
[12] The Lemma and its proof are as follows (in my translation):

Quantities proportional to their differences are continually proportional.

Let A be to $A - B$ as B to $B - C$ and C to $C - D$ etc. and there will result by division (*dividendo)* that A is to B as B is to C and C is to D etc. Q.E.D.

[13] The above diagram is the one to which MacLaurin refers earlier (see footnote (9)). The line DT is tangent to the curve at D.

remarkable phenomenon: namely, that very many people who seem to be of the opinion that virtue brings an infinite good thing after itself, nevertheless pursue vices.



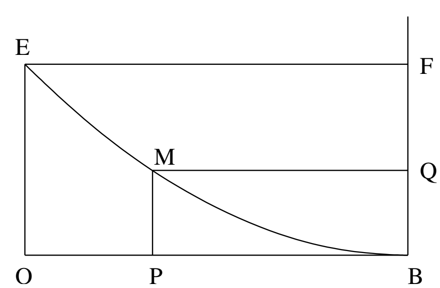
2°. The forces into different infinite good things ADQB, PNQB are as the forces at their beginnings AD, PN, and in general the force into any good thing at one distance (namely, ADBC) is to the force into the same good thing at another distance, as the force at its beginning at the first distance to the force at the beginning at the other distance: and if that good thing is set at various arithmetically proportional distances, the forces towards the entire good thing will be in geometric progression.

But this hypothesis operates with this single difficulty that the good-seeking force vanishes at infinite distance only and consequently it seems that it cannot hold in the good things of this life. Therefore let us see what other power of the good-seeking force we may select to which we set its decrease proportional.(x) We have just said that it is not unity; that it does not exceed unity is established by the fact that the curve DMC would then be of hyperbolic type, and consequently the force into the present good thing would be infinite; that it is not negative or less than zero comes from the fact that this decrease is greater the greater the force is, not the smaller; that it is not itself 0 is in accordance with the fact that this decrease varies and is not perpetually the same. Therefore it is neither 0 nor 1, neither greater than 1 nor less than 0; therefore it is between 0 and 1. What remains therefore than that we take the mean between 0 and 1, namely $\frac{1}{2}$. Therefore let $\dot{v}$ be as $v^{1/2}$ or $\sqrt{v}$. With the assumption that v is the good-seeking force and t the time of duration $=$ OP, therefore let

$$\dot{v} : \sqrt{v} :: \dot{t} : \sqrt{a}\,, \quad \text{therefore} \quad \frac{\dot{v}}{\sqrt{v}} = \frac{\dot{t}}{\sqrt{a}}$$

(Note. a is the accelerating force into the present good thing), therefore by finding the fluents

$$2a - 2\sqrt{av} = t\,, \quad \text{therefore} \quad \sqrt{v} = \frac{2a - t}{2\sqrt{a}} \quad \text{and} \quad \overline{2a - t}^{\,2} = 4av\,.$$



Therefore, if the conical parabola BME is drawn with parameter $4a = 4\,\mathrm{OE}$, focus F and axis BF, then PM will be the good-seeking accelerating force at distance OP. Hence:

1°. The good-seeking force is zero at the finite distance OB. Hence:

2°. The good-seeking force is in the duplicate ratio of the distance of the time from the point in which it vanishes.

3°. Since the area BQM is $\frac{2}{3}$ BQMP, there will be

$$\mathrm{BMP} = \frac{1}{3}\,\mathrm{BQMP} = \frac{1}{3}\,\mathrm{PM} \times \mathrm{BP} = \frac{\mathrm{BP}^3}{12\,\mathrm{OE}}\,,$$

or as BP^3, therefore

$$\mathrm{OEMP} = \frac{\mathrm{OB}^3 - \mathrm{BP}^3}{12\,\mathrm{OE}}\,.$$

3. Mathematical notes on De Viribus Mentium Bonipetis

(i) Area between the cissoid and its asymptote

The cissoid of Diocles is the curve defined by the parametric equations

$$x = \frac{2at^2}{1+t^2}\,, \quad y = \frac{2at^3}{1+t^2} \quad (t \in \mathbb{R})\,.$$

As $t \to \pm\infty$ we have $x \to 2a$, while $y \to \infty$ as $t \to \infty$ and $y \to -\infty$ as $t \to -\infty$. The line $x = 2a$ is the asymptote and the area between the cissoid and its asymptote is given by either of the integrals

$$2\int_0^{2a} y\,dx = 2\int_0^{\infty} y\frac{dx}{dt}\,dt = 16a^2\int_0^{\infty} \frac{t^4}{(1+t^2)^3}\,dt\,,$$
$$2\int_0^{\infty} (2a-x)\,dy = 2\int_0^{\infty} (2a-x)\frac{dy}{dt}\,dt = 8a^2\int_0^{\infty} \frac{t^2(3+t^2)}{(1+t^2)^3}\,dt\,,$$

whose common value can easily be shown to be $3\pi a^2$ (for example, substitute $t = \tan\theta$). Concerning this curve see [3] (especially pp. 39–40).

(ii) Area between hyperbolas of any power and their asymptotes

Here MacLaurin is referring to integrals of the form $\int_a^\infty x^{-\alpha}dx$, where $a > 0$ and α is a positive real number, in particular a positive integer. Such an integral converges if and only if $\alpha > 1$. The case $\alpha = 1$ corresponds to the rectangular hyperbola, the conic which MacLaurin excepts.

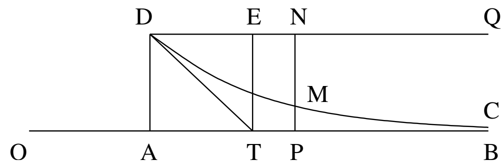
(iii) Geometric/arithmetic progressions

Here the intensity I takes the form $I = I_0e^{-kt}$ for some positive constant k. With times in arithmetic progression we have $t_n = t_0 + n\delta t$ ($n = 0, 1, \ldots$; $t_0, \delta t$ constants) and then $I_{n+1} = I_ne^{-k\delta t}$, so that the corresponding intensities form a geometric progression with common ratio $e^{-k\delta t}$.

It is at first curious that MacLaurin should refer to the curve DMC as the *common logarithm*. As an exponential it is of course a reflection of a logarithmic curve and in that sense it has the same shape. Exponential notation appears to have been introduced by Euler in the mid 1700s. Before then the idea was expressed in words: *the number whose logarithm is a given quantity*. Natural and common logarithms (base 10) are constant multiples of each other, so in describing the general shape of a curve, it does not matter which of the adjectives *common* and *natural* we use.

(iv) Areas

MacLaurin's procedure in this case for finding the area under the curve is interesting, and no doubt uses a standard rule of his time.



We have to evaluate

$$\int_a^\infty I_0 e^{-kt} dt = \frac{1}{k}\, I_0 e^{-ka}\,, \tag{α}$$

where A is the point $(a, 0)$. The tangent to the curve at the point $\mathrm{D}(a, I_0 e^{-ka})$ has gradient $-kI_0 e^{-ka}$ and equation

$$y = -kI_0 e^{-ka}(t-a) + I_0 e^{-ka}\,.$$

It therefore meets the t-axis where $t = a + \frac{1}{k}$ and so the rectangle ATED has area

$$\mathrm{AT} \times \mathrm{AD} = \frac{1}{k} \times I_0 e^{-ka}\,,$$

which is the value of the integral in (α).

In MacLaurin's subsequent calculation we have $I_{n+1} = \frac{1}{2} I_n$ with $\delta t = 1$ (day), so that $e^{-k} = \frac{1}{2}$ (see (iii) above). Over the first day the 'good thing' is

$$\int_a^{a+1} I_0 e^{-kt} dt = \frac{1}{k} I_0 \left(e^{-ka} - e^{-k(a+1)}\right) = \frac{1}{k} I_0 e^{-ka}\left(1 - e^{-k}\right) = \frac{1}{2k} I_0 e^{-ka}\,, \tag{β}$$

while over the infinite range it is as given in (α), namely twice that for the first day. For MacLaurin's discrete calculation note that the 'good things' on successive days are given by (cf. (β))

$$\frac{1}{k} I_0 e^{-k(a+n-1)}\left(1 - e^{-k}\right) = \frac{1}{2k} I_0 e^{-k(a+n-1)} \quad (n = 1, 2, \ldots)\,,$$

and so they form a geometric progression with common ratio $e^{-k} = \frac{1}{2}$.

(v) Uniform increase from zero; positive powers

In the first case we have $I = k(t-a)$ $(t \geq a)$, so that the 'good thing' as a function of t is

$$\int_a^t k(\tau - a)\, d\tau = \frac{1}{2} k(t-a)^2\,,$$

and is therefore proportional to the square of the duration. In the general case of $I = k(t-a)^n$ $(t \geq a)$ this becomes

$$\int_a^t k(\tau - a)^n\, d\tau = \frac{1}{n+1} k(t-a)^{n+1} = \frac{1}{n+1}\left(k(t-a)^n\right) \times (t-a)\,,$$

which is proportional to the product of the intensity and the duration and goes off to infinity as $t \to \infty$.

(vi) Uniform good things: Corollary 3

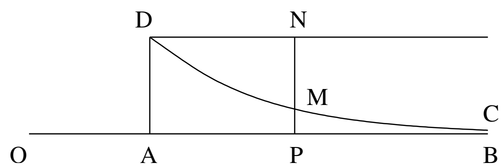
Here we require $\int_{a_1}^{t_1} k_1\, dt = \int_{a_2}^{t_2} k_2\, dt$, that is $k_1(t_1 - a_1) = k_2(t_2 - a_2)$, or equivalently for non-zero quantities

$$\frac{k_1}{k_2} = \frac{t_2 - a_2}{t_1 - a_1}\,;$$

in words: the ratio of the intensities is equal to the reciprocal of the corresponding ratio of durations.

(vii) Accelerating force

By *accelerating force* MacLaurin intends $\frac{dF}{dt}$, where $F(t)$ is the total force of attraction of the mind into the 'good thing' from its inception up to time t. He now assumes his 'good thing' to be uniform and considers various forms for $\frac{dF}{dt}$ or its derivative (Cases 1–3 below).



Since MacLaurin is generally concerned with relations of proportionality there is usually no loss of generality in taking the graph of the accelerating force through D as shown. Here and throughout I have translated MacLaurin's 'distantia temporis' literally as 'distance of time': apparently it is just the elapsed time from the start.

(viii) Case 1: reciprocally proportional to the distance of time

From $\frac{dF}{dt} = \frac{k}{t}$ ($t \geq a > 0\,;\, a = \mathrm{OA}$) we obtain

$$F(t) = \int_a^t \frac{k}{\tau}\, d\tau = k \ln \frac{t}{a}\,. \qquad (\gamma)$$

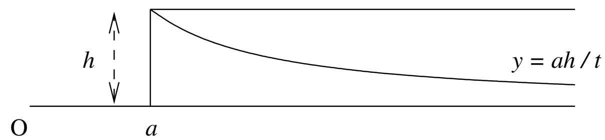
Then $F(t) - F(\tau) = k \ln(t/\tau)$. Thus, if we take a sequence of times t_n in geometric progression, so that t_{n+1}/t_n is constant, the corresponding differences of the forces will be constant; that is to say, the forces will form an arithmetic progression. This is Corollary 1, while Corollary 2 is just the fact that the integral (γ) does not converge as $t \to \infty$.

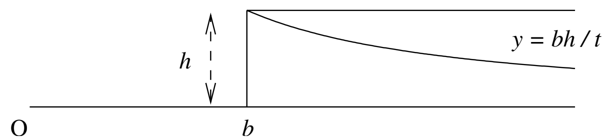
I have struggled to understand Corollary 3 but have come to the conclusion that it simply expresses the fact that, if in (γ) we replace the lower limit a by ra and the upper limit t by rt (or equivalently, replace a by ra and the duration $t - a$ by $r(t - a)$), then the value of the integral is unchanged. In the last sentence of Corollary 3 'its double' would then refer to doubling the duration.

Corollary 4 may describe the following situation where we have two 'processes' with the same starting value:

$$ah/t \quad t \in [a, \infty); \quad bh/t \quad t \in [b, \infty),$$

where a, b, h are positive constants with $a < b$.





Compare their outputs after time t:

$$\begin{aligned}\int_b^{b+t} \frac{bh}{\tau}\, d\tau - \int_a^{a+t} \frac{ah}{\tau}\, d\tau &= \int_b^{a+t} \frac{(b-a)h}{\tau}\, d\tau + \int_{a+t}^{b+t} \frac{bh}{\tau}\, d\tau - \int_a^{b} \frac{ah}{\tau}\, d\tau \\ &= (b-a)h \ln \frac{a+t}{b} + bh \ln \frac{b+t}{a+t} - ah \ln \frac{b}{a} \\ &= (b-a)h \ln \frac{a+t}{b} + bh \quad \underbrace{\ln \frac{b/t+1}{a/t+1}}_{\to \ln 1 = 0 \text{ as } t \to \infty} \quad - ah \ln \frac{b}{a} \\ &\to \infty \text{ as } t \to \infty;\end{aligned}$$

the later 'good thing' always has infinitely more force than the earlier 'good thing' no matter how close their starts.

MacLaurin rejected this model on the basis of Corollaries 3 and 4 and moved on to one of exponential decay.

(ix) Case 2: geometric decrease with arithmetic increase of time (cf. (iii))

This corresponds to $\frac{dF}{dt} = ke^{-\alpha t}$, where k, α are positive constants. The total force over the infinite interval starting at t is then

$$\int_t^{\infty} ke^{-\alpha\tau} d\tau = \frac{k}{\alpha} e^{-\alpha t}\,. \qquad (\delta)$$

As MacLaurin notes, the fluxions of the forces are as (proportional to) the forces themselves. Moreover,

$$\frac{ke^{-\alpha t_{n+1}}}{ke^{-\alpha t_n}} = e^{-\alpha(t_{n+1} - t_n)}\,,$$

so that, if the t_n are in arithmetic progression, then the corresponding (accelerating) forces are in geometric progression. Likewise, the total forces (δ) are in geometric progression.

In Corollary 1 MacLaurin uses the area result already noted (see (iv) above) to equate the total force for an infinite time interval with a related finite, uniform process. For Corollary 2 we note from (δ) that

$$\frac{\text{Total force from } t_1 \to \infty}{\text{Total force from } t_2 \to \infty} = \frac{e^{-\alpha t_1}}{e^{-\alpha t_2}} = \frac{ke^{-\alpha t_1}}{ke^{-\alpha t_2}} = \frac{\frac{dF}{dt} \text{ at } t_1}{\frac{dF}{dt} \text{ at } t_2}\,.$$

While MacLaurin was happier with this hypothesis, he did not like the fact that the forces could never vanish in finite time. This led him on to his final model which involves the second derivative.

(x) Case 3: power decrease of the accelerating force

Here we consider $v = \frac{dF}{dt}$ and set $\frac{dv}{dt} = -cv^{\alpha}$, where c is a positive constant and α is a constant. If $\alpha = 1$ we come back to Case 2; otherwise we obtain from $v^{-\alpha}\frac{dv}{dt} = -c$ that

$$\frac{v^{-\alpha+1}}{-\alpha+1} = -ct + k\,, \quad \text{and so} \quad v = ((1-\alpha)(k-ct))^{1/(1-\alpha)}\,.$$

For v to vanish at some finite time we need $1/(1-\alpha) > 0$, that is to say, $\alpha < 1$; moreover, $\alpha < 0$ makes $|\frac{dv}{dt}|$ decrease as $|v|$ increases and $\alpha = 0$ makes $\frac{dv}{dt}$ constant, neither of which MacLaurin admits. He therefore requires $0 < \alpha < 1$ and settles on $\alpha = \frac{1}{2}$ as the mean value. Thus we consider

$$v = \tfrac{1}{4}(k-ct)^2\,.$$

Setting $v(0) = a$ makes [14] $k = 2\sqrt{a}$ and choosing [15] $v'(0) = -1$ gives us $-\frac{1}{2}ck = -1$, or $c = 1/\sqrt{a}$. We now have

$$v = \frac{1}{4}\Big(2\sqrt{a} - \frac{t}{\sqrt{a}}\Big)^2\,,$$

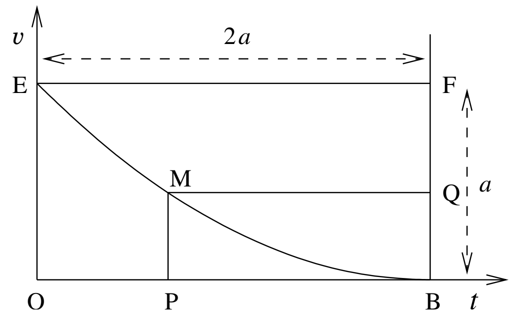
(14) The negative root will not allow v to vanish in positive time.

(15) As noted before, constants of proportionality can usually be chosen arbitrarily and it would be natural to 'normalise' by dividing by the initial value $a^{1/2}$ of $v^{1/2}$. MacLaurin does this right at the start by taking his fluxional equation as

$$\frac{\dot{v}}{\sqrt{v}} = \frac{\dot{t}}{\sqrt{a}}\,.$$

Note that he does not have the minus sign, but this is presumably implicit in $\dot{t}$ to ensure decrease.

or $4av = (2a - t)^2$, which is MacLaurin's solution. The above discussion is valid for $0 \le t \le 2a$.



The equation $4av = (2a - t)^2$ defines the parabola with vertex B at $t = 2a$, $v = 0$ and focus F at $t = 2a$, $v = a$; it crosses the v-axis at E, where $t = 0$, $v = a$.

For the upper part of the standard parabola $y^2 = 4ax$ we have

$$\int_0^c y\,dx = \int_0^c 2\sqrt{a}\sqrt{x}\,dx = \tfrac{4}{3}\sqrt{a}\,c^{3/2} = \tfrac{2}{3}c \times 2\sqrt{ac} = \tfrac{2}{3}\,\mathrm{OC} \times \mathrm{CT}\,,$$

where O is the origin (vertex), C is the point $(c, 0)$ $(c \ge 0)$ and T is the point $(c, 2\sqrt{ac})$ on the curve. MacLaurin applies this geometric result to get the (curvilinear) area BQM equal to $\frac{2}{3}$ BQMP and then the area

$$\mathrm{BMP} = \mathrm{BQMP} - \tfrac{2}{3}\,\mathrm{BQMP} = \tfrac{1}{3}\,\mathrm{BQMP} = \tfrac{1}{3}\,\mathrm{PM} \times \mathrm{BP}\,.$$

Moreover,

$$\mathrm{PM} = \frac{1}{4a}(2a - t)^2 = \frac{\mathrm{BP}^2}{4 \times \mathrm{OE}} \quad \text{so} \quad \mathrm{BMP} = \frac{\mathrm{BP}^3}{12 \times \mathrm{OE}}\,.$$

This in turn gives

$$\mathrm{BEO} = \frac{\mathrm{BO}^3}{12 \times \mathrm{OE}} \quad \text{and} \quad \mathrm{OEMP} = \mathrm{BEO} - \mathrm{BMP} = \frac{\mathrm{OB}^3 - \mathrm{BP}^3}{12 \times \mathrm{OE}}\,.$$

4. The second part of MacLaurin's paper

As noted in the Introduction the second part of MacLaurin's manuscript is concerned with Propositions 21 and 22 and their Scholium from Book II of Newton's *Principia* ([13], [14]). These occur in a section on density and compression of fluids and hydrostatics. To help show the connection with MacLaurin's results I begin with a translation of these propositions and of Newton's definition of a fluid. This is followed by my translation of the second part of MacLaurin's manuscript. Finally I have appended a short commentary on its contents.

Definition. *A fluid is any body whose parts yield to any applied force, and moves easily within itself by yielding.*

Proposition 21. *Let the density of a certain fluid be proportional to the compression and let its parts be drawn downwards by a centripetal force reciprocally proportional to their distances from the centre: I say that, if those distances are taken to be continually proportional, the densities of the fluid at the same distances will also be continually proportional.*

Proposition 22. *Let the density of a certain fluid be proportional to the compression and let its parts be drawn downwards by gravity reciprocally proportional to the squares of their distances from the centre: I say that, if the distances are taken in musical progression,* (16) *the densities of the fluid at these distances will be in geometric progression.*

Each proposition has a corollary showing how the density at any point can be obtained if the density is given at two points; a hyperbola is used in the corollary to Proposition 21 (cf. MacLaurin's §IV, §V below). In the Scholium Newton indicates what happens under other assumptions about the gravity, some of which are implicit in MacLaurin's general result. Here is a translation of the second part of Maclaurin's manuscript.

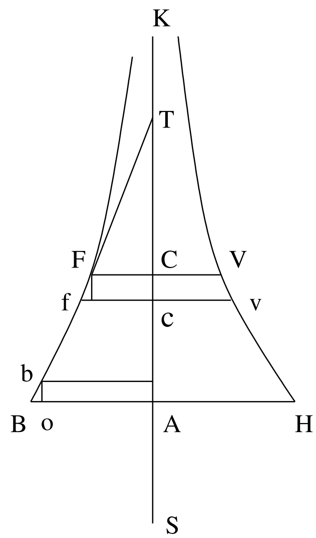
Second proposition

Everywhere about the point S let any fluid lie, gravitating to it with centripetal forces which are at the distances SC, Sc, SA as CV, cv, AH, ordinates of the curve HVK. Let that fluid be compressed by forces which are as FC, fc, AB at the same distances SC, Sc, SA: let B be a point in which the curve cuts AB at an angle of 45°, where AB = AS. Let

$$\mathrm{AS} = e\,, \quad \mathrm{AH} = a\,, \quad \mathrm{CV} = c\,, \quad \mathrm{AC} = x\,, \quad \mathrm{FC} = F\,.$$

Let the density of the fluid at A = b, at C = D.

(16) See the *Commentary* below (after equation (9)).



Since the increment of the compressing force is the weight of the increment of the superincumbent fluid and since the weight of this increment is as its quantity of material and the accelerating force combined, or as the density of its mass and the accelerating force, i.e., as $D \times c \times -\dot{x}$, the increment of the compressing force or $\dot{F}$ will be as $-Dc\dot{x}$ and Bo (its increment at distance SA) will be as $ab \times$ ob, therefore

$$\dot{F} \, : \, \mathrm{Bo} \, :: \, -Dc\dot{x} \, : \, ab \times \mathrm{bo}\,,$$

therefore

$$\dot{F} = \frac{-Dc\dot{x} \times \mathrm{Bo}}{ab \times \mathrm{bo}} = \ (\text{since Bo} = \text{bo})\ \frac{-Dc\dot{x}}{ab}\,.$$

Corol. I. The specific gravity of the fluid at C is to its specific gravity at A as FC to the subtangent CT. For the specific gravity of the fluid is the weight of a given amount, and so compositely as the density and the accelerating force: and consequently

$$\text{the specific gravity at C is to that at A} :: \, Dc \, : \, ab \, :: \, \dot{F} \, : \, \dot{x} \, :: \, \mathrm{FC} \, : \, \mathrm{CT}\,.$$

Corol. II. Since

$$\mathrm{CT} = \frac{-\mathrm{FC} \times \dot{x}}{\dot{F}} = \frac{-Fab}{Dc}\,,$$

CT is negative, and so BF is convex to the axis SK.

§II. Let the density be supposed proportional to the compression or $F = D$ and

$$\dot{F} = \dot{D} = \frac{-Dc\dot{x}}{ab} = \ (b = e)\ = \frac{-Dc\dot{x}}{ae}\,,$$

and *

$$\mathcal{F}\,\frac{-\dot{D}}{D} = \mathcal{F}\,\frac{c\dot{x}}{ae}\,, \quad \text{therefore} \quad \frac{\mathrm{AHVC}}{ae} = \mathrm{L}, D\,.$$

* Note. Here $\mathcal{F}$ denotes the fluent quantity. (MacLaurin's footnote)

(Note. L denotes the logarithm.)

Corol. I. If the areas AHVC are taken in arithmetic progression, the densities will be in geometric progression.

Corol. II. The subtangents of the curve BF are reciprocally as the centripetal forces, for

$$\mathrm{CT} = \frac{D\dot{x}}{-\dot{D}} = \frac{ae}{c}\,, \quad \text{or as} \quad \frac{1}{c}\,.$$

Corol. III. If $F = D$ the ordinates of the curve BF will be as the densities.

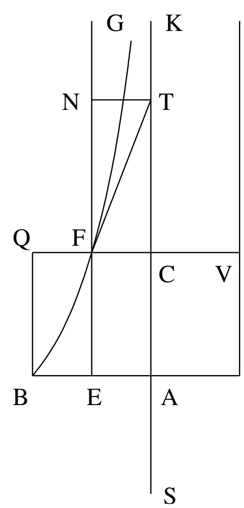
§III. Ex I. Let the centripetal force be constant or $c = a$ and

$$\dot{D} = \frac{-Dc\dot{x}}{ae} = \frac{-D\dot{x}}{e} \quad \text{and} \quad \frac{-\dot{D}}{D} = \frac{\dot{x}}{e}\,,$$

from which it is clear that the curve BF is logarithmic, drawn with asymptote SK and subtangent e.

Corol. I. If the distances are taken in arithmetic progression, the densities will be in geometric progression.

Corol. II. The specific gravities of the fluid will be in the same progression (namely, geometric), for, on account of the given centripetal force, the specific gravity is as the density.



Corol. III. The quantity of superincumbent fluid in any column CFGK, however much it may fill up the infinite space, is nevertheless finite. For the quantity above the base FC is equal to the homogeneous quantity which the tube FCTN of density CF would contain. And the quantity between A and C above the base AB is equal to the homogeneous quantity BEQF of density AB.

Corol. IV. If the distances are taken in arithmetic progression, the superincumbent quantities of fluid will be in geometric progression, for

$$\mathrm{GFCK} = \mathrm{FC} \times \mathrm{TN}\,, \quad \text{or as} \quad \mathrm{FC}\,.$$

§IV. Ex II. Let the centripetal force be reciprocally as the distance, or $c = \dfrac{ae}{e+x}$, and so

$$-\dot{D} = \frac{De\dot{x}}{e \times \overline{\overline{e+x}}} = \frac{D\dot{x}}{e+x}\,,$$

therefore

$$-e\dot{D} - x\dot{D} = D\dot{x}\,, \quad \text{and} \quad -e\dot{D} = D\dot{x} + x\dot{D}\,,$$

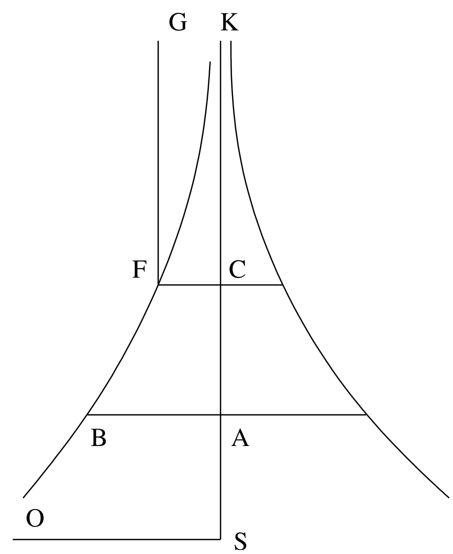
therefore by finding the fluent quantities

$$e^2 - eD = Dx\,, \quad \text{and} \quad e^2 = Dx + eD = D \times \overline{e+x}\,,$$

from which it is clear that the curve BF is the equilateral hyperbola drawn with centre S and asymptotes SK, SO.

Corol. I. If the distances are taken in geometric progression, the densities, the compressing forces and the centripetal forces will be in the same progression and they are all mutually proportional.

Corol. II. The densities are reciprocally as the distances.



Corol. III. The total quantities of material in the infinite column CKGF are infinite. For if the distances are taken in geometric progression, the quantities of material in a given column will be in arithmetic progression.

Corol. IV. The specific gravities of the fluid are reciprocally as the squares of the distances, for they are as $Dc = \dfrac{1}{\overline{\overline{e+x}}|^2}$.

Schol. And in this hypothesis this is the thing most deserving to be noted, that in it alone the density (and so the compressing force or even the specific gravity of the fluid) can be as some power of the distance if $F = D$. For if $D = \dfrac{e^{m+1}}{\overline{e+x}|^{m}}$ there will be

$$\dot{D} = \frac{-me^{m+1}\dot{x}}{\overline{e+x}|^{m+1}} = \frac{-Dc\dot{x}}{ae} = \frac{-e^{m+1}c\dot{x}}{ae \times \overline{e+x}|^{m}}\,,$$

$$\text{therefore} \quad \frac{m\dot{x}}{e+x} = \frac{c\dot{x}}{ae}\,, \quad \text{therefore} \quad c = \frac{mae}{e+x}\,,$$

or reciprocally as the distance.

§V. Universally let

$$c = \frac{ae^{n}}{\overline{e+x}|^{n}} \quad \text{and} \quad -\dot{D} = \frac{De^{n}\dot{x}}{e \times \overline{e+x}|^{n}}\,.$$

Therefore

$$\mathcal{F}\,\frac{-\dot{D}}{D} = \frac{-e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}} + \frac{e^{n-1}}{\overline{n-1} \times e^{n-1}}$$

$$= \frac{\overline{n-1} \times \overline{e+x}|^{n-1} - \overline{n-1} \times e^{n-1}}{\overline{n-1}|^{2} \times \overline{e+x}|^{n-1}} = \frac{\overline{e+x}|^{n-1} - e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}}$$

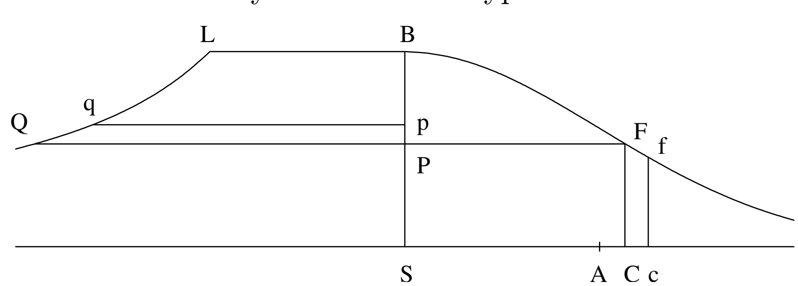
$$= -\mathrm{L}, D\,.$$

Corol. I. Therefore if the $1-n$ power of the distance is taken in arithmetic progression, the densities will be in geometric progression.

Corol. II. Let the centripetal force be directly as the distance and let $n = -1$, therefore $1 - n = 2$, and so, if the squares of the distances are taken in arithmetic progression, the densities will be in geometric progression and the equation of the curve BF will be

$$\frac{-\dot{D}}{D} = \frac{\overline{e+x} \times \dot{x}}{e^{2}}\,,$$

which can be constructed thus by means of the hyperbola.[17]



[17] MacLaurin's diagram shows the constructed curve as concave up. I believe it should be as I have drawn it. See discussion below (equation (7)).

Let the equilateral hyperbola LQ be drawn with centre S and let SB $= e =$ SA $=$ LB. Let us take SC $= \sqrt{2\,\mathrm{BLQP}}$ and FC $=$ SP; the curve passing through all points found in this way will be BF, the curve requiring to be constructed. For

$$\mathrm{QP} = \frac{\mathrm{SB}^2}{\mathrm{SP}} = \frac{e^2}{D} \quad \text{and} \quad \mathrm{QPpq} = \frac{-\dot{D}e^2}{D} = \overline{e+x} \times \dot{x} = \mathrm{SC} \times \mathrm{Cc}\,,$$

therefore

$$\frac{-\dot{D}}{D} = \frac{\overline{e+x} \times \dot{x}}{e^2}\,,$$

which was the equation requiring to be constructed.

Note. The subtangent of this curve is reciprocally as the distance.

Corol. III. Let the centripetal force be reciprocally as the square of the distance, or $c = \dfrac{ae^2}{\overline{e+x}|^2}$. There will be $n = +2$ and $1 - n = -1$, therefore, if quantites reciprocally proportional to the distances are taken in arithmetic progression, or the distances themselves in musical progression, the densities will be in geometric progression; and the equation of the curve will be

$$\frac{-\dot{D}}{D} = \frac{e\dot{x}}{\overline{e+x}|^2}\,.$$

This can also be easily constructed by means of the hyperbola.

VI. Now let the compressing force be supposed proportional to some power of the density, or

$$F = D^m \quad \text{and} \quad \dot{F} = mD^{m-1}\dot{D} = \frac{Dc\dot{x}}{ae}\,.$$

Therefore

$$mD^{m-2}\dot{D} = \frac{c\dot{x}}{ae} \quad \text{and} \quad \frac{mD^{m-1}}{m-1} = \mathcal{F}\,\frac{c\dot{x}}{ae}\,,$$

or D^{m-1} is as the area AHVC.

Further let $c = \dfrac{ae^n}{\overline{e+x}|^n}$ and there will be $mD^{m-1}\dot{D} = \dfrac{e^{n-1}\dot{x}}{\overline{e+x}|^n}$, therefore

$$\frac{m}{m-1}D^{m-1} = \frac{-e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}} \quad \text{(if } b \text{ is the density at distance } e\text{)}$$

$$+ \frac{mb^{m-1}}{m-1} + \frac{1}{n-1}\,,$$

$$\frac{m}{m-1}D^{m-1} = \frac{mb^{m-1}}{m-1} + \frac{1}{n-1} - \frac{e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}}\,.$$

Corol. I. If $m = n$ the curve whose ordinates are as the densities will be a conical hyperbola and the density will be reciprocally as the distance.

Commentary. MacLaurin first sets up the fluxional equation

$$\dot{F} = -\frac{Dc\dot{x}}{ab}\,, \tag{1}$$

where F (compressing force), D (density), c (centripetal force) are functions of x (distance from a specially chosen initial point A) and a, b are the values of c, D respectively at A ($x = 0$). MacLaurin's instruction to 'let B be a point in which the curve cuts AB at an angle of 45°, where AB = AS' is equivalent to requiring

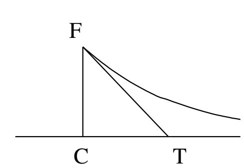
$$F(0)\,(=\mathrm{AB})\;= e\,, \qquad F'(0) = -1\,. \tag{2}$$

At first these requirements seem arbitrary and restrictive, but I believe that they involve simply changing vertical and horizontal scales or physical units: having chosen A, put $x = \alpha\xi$, $y = \beta\eta$ in $y = F(x)$ to get

$$\eta = \frac{1}{\beta}F(\alpha\xi)\,, \qquad \frac{d\eta}{d\xi} = \frac{\alpha}{\beta}F'(\alpha\xi)\,;$$

clearly, we may choose α, β to make $\eta = e$ and $\frac{d\eta}{d\xi} = -1$ when $\xi = 0$ (provided $F(0) \neq 0$, $F'(0) \neq 0$). The changes will not affect the proportional relationships in which MacLaurin is interested and so there is no loss of generality in assuming (2). Not being an expert in fluids I will not comment further on the derivation of equation (1).

Maclaurin uses the term *subtangent* in several places. This refers to CT as shown in the diagram, where the tangent to the curve $y = f(x)$ at the point F meets the axis in T and C is the projection of F on the same axis. A simple calculation shows that the signed length CT is given by



$$\mathrm{CT} = -\frac{f(x_F)}{f'(x_F)}\,. \tag{3}$$

This is equivalent to the expression used by MacLaurin in Corollary II of the first section.

Under the assumption that D is proportional to F, which MacLaurin takes in §II as $D = F$,[18] implying that b is equal to the distance AS denoted by e,[19] equation (1) becomes

$$\frac{\dot{D}}{D} = -\frac{c\dot{x}}{ae} \,. \tag{4}$$

We therefore have

$$\ln D = -\frac{1}{ae}\int_0^x c\,dx + \ln e \,, \tag{5}$$

and

$$D = e\exp\big(-\frac{1}{ae}\int_0^x c\,dx\big) \,. \tag{6}$$

Equation (5) is MacLaurin's equation

$$\frac{\mathrm{AHVC}}{ae} = \mathrm{L}, D$$

in §II if we interpret L, D as $\ln(D/e)$.

In §III MacLaurin takes c to be constant ($c = a$), a case which Newton says in his Scholium was investigated by Halley. Here we have from equation (6)

$$D = e\exp\big(-\frac{x}{e}\big) \,,$$

and from equation (3) we see that in this case the subtangent is always e as MacLaurin states. Note that in Corollary III MacLaurin is using the area result discussed in Section 3 (iv).

The case of c varying 'reciprocally as the distance' is considered in §IV. Now

$$c = \frac{ae}{e+x} \qquad (\text{recall: } c(0) = a) \,,$$

and from equation (6)

$$D = e\exp\big(-\int_0^x \frac{1}{e+x}\,dx\big) = \frac{e^2}{e+x} \,,$$

so that we have a rectangular hyperbola

$$D(e+x) = e^2 \,.$$

---

(18) This may be thought of as another change of units and again we note that, since MacLaurin is only concerned with proportional relationships, it is of no consequence.

(19) N.B. This is not to be confused with the number $e = \exp 1$.

The general power case

$$c = \frac{ae^n}{(e+x)^n} \qquad (n \neq 1)$$

is discussed in §V. We now have from equation (6)

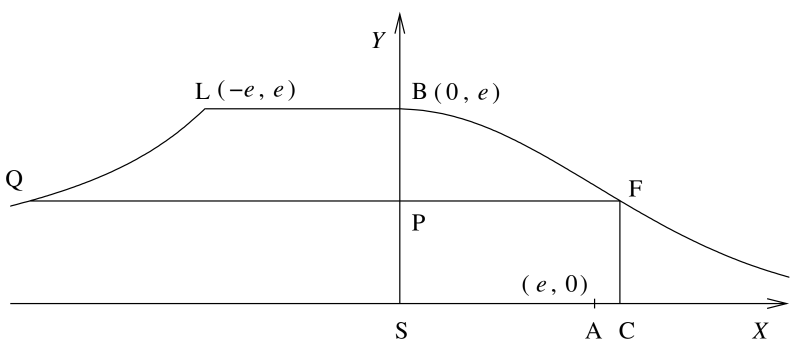
$$D = e\exp\Big(-e^{n-1}\int_0^x \frac{1}{(e+x)^n}\,dx\Big) = e\exp\Big(\frac{e^{n-1}}{n-1}\big(\frac{1}{(e+x)^{n-1}} - \frac{1}{e^{n-1}}\big)\Big)$$

$$= e\exp\Big(\frac{e^{n-1}-(e+x)^{n-1}}{(n-1)(e+x)^{n-1}}\Big). \tag{7}$$

In Corollary II of §V MacLaurin discusses the case $n = -1$, for which we obtain from equation (7)

$$D = e\exp\Big(-\frac{1}{2e^2}(x+e)^2 + \frac{1}{2}\Big) = e\exp\Big(\frac{1}{2}\Big)\exp\Big(-\frac{1}{2e^2}(x+e)^2\Big). \tag{8}$$

Although his lack of notation prevented him from providing an explicit formula here, MacLaurin nevertheless attempted to construct the solution curve.



Introducing coordinate axes in MacLaurin's diagram as shown, we have for his hyperbola (LQ) the equation $XY = -e^2$ and, if Q is the point (X, Y) on this curve ($X \leq -e$), the area BLQP is

$$\left|\int_Y^e -\frac{e^2}{Y}\,dY\right| = e^2 \ln\frac{e}{Y}.$$

The point F is then, according to MacLaurin's recipe,

$$\Big(\sqrt{2}\,e\Big(\ln\frac{e}{Y}\Big)^{1/2}, Y\Big),$$

and its locus is therefore the curve with equation

$$X = \sqrt{2}\,e\Big(\ln\frac{e}{Y}\Big)^{1/2},$$

from which we obtain

$$\ln\frac{e}{Y} = \frac{1}{2e^2}X^2, \quad \text{and therefore} \quad Y = e\exp\Big(-\frac{1}{2e^2}X^2\Big).$$

In MacLaurin's notation $X = x + e$, so we have apparently

$$D = e\exp\Big(-\frac{1}{2e^2}(x+e)^2\Big)\,. \tag{9}$$

This is equation (8) without the constant factor $\exp(1/2)$. Of course, the value of $\dot{D}/D$ is the same in both cases and we have seen that, since MacLaurin was interested in proportional relationships, the actual constants of proportionality are largely irrelevant. On the other hand, MacLaurin may simply have overlooked the fact that he wanted $D = e$ when $x = 0$ and not when $X = 0$.

Corollary III of §V deals with the case $n = 2$, for which (from equation (7))

$$D = e\exp\Big(\frac{-x}{e+x}\Big) = e\exp\Big(\frac{e}{e+x} - 1\Big)\,. \tag{10}$$

Also in this corollary MacLaurin takes the distances in *musical progression*, a condition which is imposed in Newton's Proposition 22 and is also known as *harmonic progression*. It is that, if $d_1 < d_2 < d_3$, then

$$\frac{d_3}{d_1} = \frac{d_3 - d_2}{d_2 - d_1}\,,$$

in which case

$$\frac{1}{d_3} - \frac{1}{d_2} = \frac{d_2 - d_3}{d_2 d_3} = \frac{d_1 - d_2}{d_1 d_2} = \frac{1}{d_2} - \frac{1}{d_1}\,;$$

thus the reciprocals are in arithmetic progression and conversely. With this observation MacLaurin's assertion that, if the distances (i.e. $x+e$) are taken in musical progression, then the densities are in geometric progression follows immediately from equation (10) and the properties of the exponential function.

Finally, in §VI MacLaurin replaces the hypothesis $F = D$ with $F = D^m$ $(m \neq 1)$, some particular cases of which are mentioned in Newton's Scholium. Equation (1) now becomes

$$mD^{m-1}\dot{D} = -\frac{Dc\dot{x}}{ab}\,, \quad \text{or} \quad mD^{m-2}\dot{D} = -\frac{c\dot{x}}{ab}\,. \tag{11}$$

At this point MacLaurin appears to go wrong: the $-$ does not appear and b and e are still taken as equal – perhaps we should now have $e = b^m$. We obtain from equation (11) (with $e = b^m$)

$$\frac{m}{m-1}D^{m-1} = -\frac{1}{ab}\int_0^x c\,dx + \frac{m}{m-1}b^{m-1} = \frac{1}{b}\Big(\frac{me}{m-1} - \frac{1}{a}\int_0^x c\,dx\Big)\,.$$

Then with $c = \dfrac{ae^n}{(e+x)^n}$ $(n \neq 1)$ this becomes

$$\frac{m}{m-1}D^{m-1} = \frac{1}{b}\Big(\frac{me}{m-1} + \frac{e^n}{(n-1)(e+x)^{n-1}} - \frac{e}{n-1}\Big)\,. \tag{12}$$

In the Corollary we have $m = n$ in which case equation (12) becomes

$$bmD^{m-1} = (m-1)e + \frac{e^m}{(e+x)^{m-1}}\,. \tag{13}$$

Clearly, neither this nor MacLaurin's version represents a conical hyperbola unless $m = 2$, so what does MacLaurin mean when he appears to assert that his equation does so in general? If we put

$$X = D^{m-1} - \frac{(m-1)e}{bm}\,, \quad Y = (e+x)^{m-1}\,,$$

we obtain from equation (13)

$$XY = \frac{e^m}{bm}\,,$$

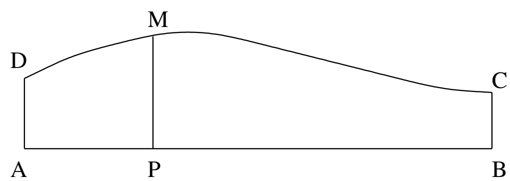
which does define a rectangular hyperbola; perhaps this is what he had in mind. Alternatively, he may have read erroneously the constants in his version of equation (12) as cancelling out when $m = n$; then he would have had

$$D(e+x) = \text{ constant}\,.$$

5. MacLaurin's Latin text

De Viribus Mentium Bonipetis

Mentes nostrae omnem bonitatis apparentis particulam in bono quovis appetentes, ad ipsum bonum necessario feruntur, vi quae est summa virium quibus singulae ejus particulae appetuntur. Sed si aequalibus viribus aequales partes appetantur, sive eadem sit omnium vis bonipeta acceleratrix; summa virium quibus boni cujusvis partes appetuntur, erit ut summa partium sive ut quantitas bonitatis in isto bono: & proinde, vires quibus mentes nostrae in diversa bona feruntur, sunt (caeteris paribus) ut quantitates bonitatis in istis bonis. Ut igitur istarum virium rationes assequamur, ipsorum bonorum quantitates sunt prius eruendae.



Repraesentet $\overline{\text{igr}}$ linea recta AB boni durationem, ad cujus singula puncta P erigantur normales PM, quae sint ut boni intensiones (i.e. bonitates instantaniae) ad finem temporis AP, & si AD sit intensio qua bonum incipit, & BC qua desinit existere, tunc bonum totale durationis AB erit ut area ADCB. Nam bonum totale, est summa bonitatum instantaniarum sive ut summa ordinatarum PM, atque adeo ut area curvilinea ADMCB vel si intensio PM appelletur I & bonum quod praeteritum est tempore AP dicatur B, & tempus AP t erit ex ipsa intensionis def:

$$\frac{\dot{B}}{\dot{t}} = I\,, \quad \dot{B} = I\dot{t} \quad \& \quad B = \mathcal{F}\, I\dot{t} = \text{ADMP}\,,$$

ergo bonum durationis AB = ADCB. Si pro intensionibus & durationibus realibus PM & AP, substituantur intensiones & durationes apparentes, ope hujus prop. inveniri potest boni quantitas apparens.

Not: haec malis facile applicari posse atque alia etiam quae sequuntur.

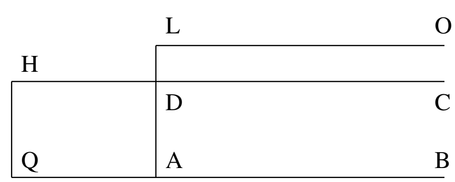
Ope hujus prop. boni vel mali cujusvis intensio maxima vel minima, per vulgares maximorum & minimorum methodos inveniuntur, nam ubi PM $\bar{\text{e}}$ ordinata maxima vel minima intensio est ejusdem ordinis. Quod si bonum non sit continuum, sed aliqua intercedat temporis distantia inter diversas ejus partes, respici potest tanquam aggregatum diversorum minorum bonorum quorum singulae per hanc prop. inventae & simul additae id conficiunt.

Porro hinc patet, plurima bona quorum durationes sunt infinitae, ipsa esse finita; plurimae enim extant areae curvilineae in infinitum porrectae sive infinité longae, quae tamen sunt finitae. Hujusmodi est area inter cissoidem ejusque asymptoton atque inter hyperbolas cujuscunque dignitatis (excepta sola conica) eorumque asmyptotes. Sed specialius si boni cujusvis intensiones decrescant in progressione geometrica dum tempora crescunt in arithmetica erit in hoc casu curva DMC logarithmica vulgaris (vide Figur. pag. 8) [20] & area infinita ADCB = ATED, supp. quod DT tangit curvam in D. Bonum igitur hujusmodi infinitae durationis AB aequale est uniformi durationis AT, intensionis AD constantis. Eodem modo si bonum quodvis uno die perceptum datam habeat quantitatem sequenti ejus dimidiam tantum partem & ita perpetuo decrescat ut bonum uno die perceptum sit dimidium precedentis & duplum sequentis: bonum inquam totale hujusmodi per infinitos dies continuatum est tantum duplum boni quod primo die percipiebatur, quod patet ex eo quod

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \ \&\text{c} := 2\,.$$

Si boni intensiones a nihilo exordium sumentes uniformiter crescant, erunt intensiones durationibus directé proportionales, & ipsa bona ut durationum quadrata. Quod si intensio sit ut durationis praeteritae potestas quaevis positiva, erunt tamen perpetuo bona in ratione composita durationum & intensionum; & proinde si illorum durationes sint infinitae erunt ipsa bona infinité infinita.

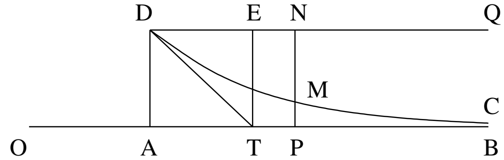
Bona etiam uniformia, quorum intensiones sunt invariabiles, sunt ut durationes & intensiones conjunctim; quoniam fig ADCB nunc evadit parallelogrammum.



1°, Hinc bonum hujusmodi cujus duratio est infinita erit ipsum infinitum. Hinc

2°, infinité magis praestat hujusmodi infinité durantis boni intensionem vel maximé exigua quantitate augere quam ejus durationem quam maxime prolongare; nam addendo DL intensioni AD boni infiniti ADCB, augmentum infinitum DLOC ei adjungimus; addendo autem AQ durationi infinitae AB fit incrementum finiti boni AQHD.

[20] The reference is to p. 8 of MacLaurin's manuscript. Here the appropriate diagram is on p. 32, but is reproduced below for convenience.

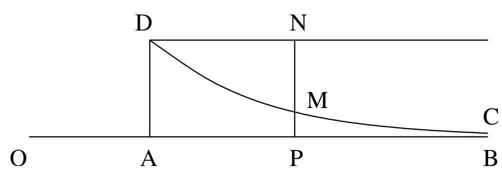


Bonorum igitur virorum de miseriis hujus vitae querelas debet perimere consideratio incrementi quod exinde intensioni faelicitatis futurae aeternae accedit, quo (forte) fit ut eorum faelicitas tota simul sumpta major sit quam si ab initio eorum existentiae exordium sumpsisset, & nunquam cecidisset homo. Intensionem autem augeri ex eo satis patere videtur quod major sit gratitudo & quod quamplurimae virtutes exerceantur quibus vix ullus esset locus si omnes homines innocentes sine ullis malis aut infortuniis vitam agerent.

3°, Ut duo hujusmodi bona sint aequalia debent intensiones esse durationibus reciproce proportionales.

Haec de bonorum mensura in genere de quibus nonnihil specialius postea erit dicendum.

Hactenus vim bonipetam acceleratricem sive vim qua data pars boni appetitur constantem ac ubique eandem supposuimus. Eam autem pro diversa temporis distantia variari aliisque ex causis experientia docet. Supponatur bonum esse uniforme. Si in datis PN sumantur PM viribus bonipetis acceleratricibus ad distantias OP proportionales, erit area ADMP ut vis qua mens urgetur ad bonum APND.



Supponatur igitur imprimis vis bonipeta distantiae temporis reciproce esse proportionalis, eritque curva DMC hyperbola Apollonia ad centrum O asymptotes OD, [21] OB descripta. Hac in hyp.

1°, Si durationes boni auctae distantia OA sumantur in progressione geometrica, erunt vires quibus mens tendit ad illa bona in progressione arithmetica & vires acceleratrices in progressione geometrica.

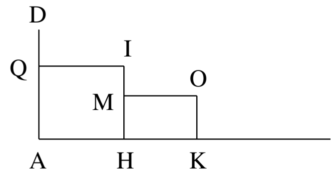
2°, Bonum infinitae durationis vi infinita mentem ad se attrahit nam area hyperbolica ADCB est infinita.

3°, Secundum hanc hypothesin bona quae sunt ut distantiae temporum mentis ab iis, aequaliter appetuntur. Sic ut bona quae sunt ut numeri 1, 2, 3, 4, 5, 6 ad temporum distantias quae respectivé sunt ut 1, 2, 3, 4, 5, 6 aequaliter appetentur v.g. bonum quodvis hora ab hac distans ejusque duplum duas post horas obtinendum aequaliter secundum hanc hyp. appeterentur.

4°, Vis in bonum praesens infinité major est vi in idem bonum vel minima distantia remotum. Nec unquam ad ullam distantiam non infinitam est nulla.

[21] In MacLaurin's diagram OD is certainly not an asymptote. Perhaps he means the limiting position of OD as A $\to$ O.

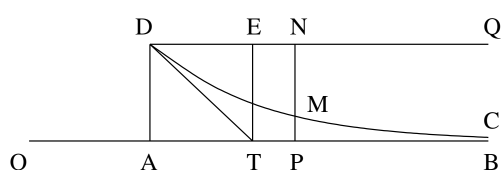
Atque haec duo posteriora Corollaria me movent ut existimem hanc hyp. non esse veram.



Urgeatur mens nostra ad bonum quodvis presens vi ut AD; ad distantiam autem AH, ut HI. Nunc supponatur vi ut AQ (= HI) presens bonum appetere & ad distantiam AH id appetet vi quae sit ad AQ :: HI : AD, sive ut HM. Sed si HK sumatur = AH ob easdem rationes sicut AQ (= HI) minuobatur in MH, nunc HI evadet OK (= MH); ita ut AD, HI, KO sint continue proportionales; & sic deinceps unde probabile est vires bonipetas acceleratrices decrescere in progressione geometrica si tempora crescant in arithmetica.

Secundum hanc hypothesin decrementa virium sive fluxiones erunt ut ipsae vires (patet ex Lemm I$^{\text{mo}}$ Lib 2$^{\text{di}}$ Princip. Neutoni). Quae hyp. malitiae ac diligentiae Temptatoris est satis consentanea, qui verisimiliter majorem adhibet operam ad vim cohibendam quo major est; unde nonnihil verisimile est Satanam conari hanc legem in mentibus nostris stabilire saltem injicere dum bona vitae futurae consideramus.

Haec hypothesis experientia nonnihil confirmatur: nam excessus vis in bonum hora ab hoc instante distans, supra vim in bonum duas post horas obtinendum, major est excessu vis in bonum anno remotum, supra vim in bonum post annum & horam obtinendum. De hac differentia non sumus adeo soliciti quam de illa unde patet vim bonipetam acceleratricem non uniformiter decrescere, ita ut quaevis temporis particula eandem quantitatem ex ea auferat. Sed potius celerius decrescere dum major est, & tardius quum minor evadit; ita ut decrementum vis sit ut ipsa vis vel aliqua ejus potestas.

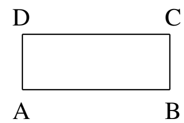


Hac igitur posita: [22]

1°, Quoniam area ADCB = ADET erit vis in bonum infinitum ADEQBA aequalis vi in bonum ATED & vis AD continuetur uniformis propter tempus AT. Vis $\overline{\text{igr}}$ qua bonum infinitum secundum hanc legem appetitur, est finita & temporariis & finitis bonis aequari, vel etiam vinci potest. Atque hinc prodiit modus solvendi phaenomenon insigne:

[22] The above diagram is the one to which MacLaurin refers earlier as being on page 8. The line DT is tangent to the curve at D.

$\overline{\text{nim}}$. quod plurimi qui virtutem bonum infinitum post se trahere opinare videntur, vitia tamen sequuntur.



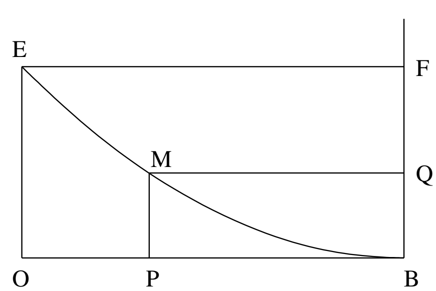
2°, Vires in diversa infinita bona ADQB, PNQB sunt, ut vires in eorum principiis AD, PN, & in universum vis in bonum quodvis ad unam distantiam ($\overline{\text{nim}}$. ADBC) est ad vim in idem ad aliam distantiam, ut vis in principio ejus ad primam distantiam ad vim in principio ad alteram distantiam: & si illud bonum disponatur in distantiis diversis arithmeticé proportionalibus, erunt vires versus totum bonum in progressione geometrica.

Sed hac sola difficultate laborat haec hypothesis, quod vis bonipeta ad solam distantiam infinitam evanescat & proinde in bonis hujus vitae obtinere non posse videtur. Videamus igitur quam aliam vis bonipetae potentiam eligamus cui proportionalem ejus decrementum statuamus. Eam non esse unitatem modo diximus, eam ō superare unitatem, ex eo constat, quod tum curva DMC esset hyperbolici generis, & proinde vis in bonum praesens esset infinita, eam non esse negativam seu minorem 0, ex eo constat, quod decrementum hoc sit majus quo major sit vis non quo minor, eam non esse ipsum 0, ex eo constat quod varietur hoc decrementum, nec perpetuo eadem sit. Neque $\overline{\text{igr}}$ est 0, nec 1, nec major 1, nec minor 0, ergo est inter 0 & 1, quid restat igitur quam ut medium inter 0 & 1 sumamus, $\overline{\text{nim}}$. $\frac{1}{2}$. Sit igitur $\dot{v}$ ut $v^{1/2}$ sive $\sqrt{v}$. Supp. quod v est vis bonipeta & t tempus durationis = OP, sit igitur

$$\dot{v} : \sqrt{v} :: \dot{t} : \sqrt{a} \quad \text{ergo} \quad \frac{\dot{v}}{\sqrt{v}} = \frac{\dot{t}}{\sqrt{a}}$$

(N. a est vis acceleratrix in praesens bonum), ergo fluentes inveniendo

$$2a - 2\sqrt{av} = t \quad \text{ergo} \quad \sqrt{v} = \frac{2a-t}{2\sqrt{a}} \quad \& \quad \overline{2a-t}^{\,2} = 4av\,.$$



Ergo si parametro $4a = 4\,$OE foco F axe BF, describatur parabola conica BME, erit PM vis bonipeta acceleratrix ad distantiam OP. Hinc

1°, vis bonipeta ad distantiam finitam OB est nulla. Hinc

2°, est vis bonipeta in duplicata ratione distantiae temporis a puncto in quo evanescit.

3°, Quoniam area BQM est $\frac{2}{3}$ BQMP erit

$$\mathrm{BMP} = \frac{1}{3}\,\mathrm{BQMP} = \frac{1}{3}\,\mathrm{PM} \times \mathrm{BP} = \frac{\mathrm{BP}^3}{12\,\mathrm{OE}}\,,$$

sive ut BP^3, ergo

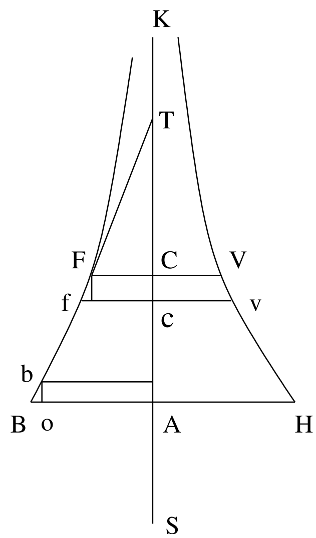
$$\mathrm{OEMP} = \frac{\mathrm{OB}^3 - \mathrm{BP}^3}{12\,\mathrm{OE}}\,.$$

Prop. altera

Circa punctum S ubicunque jaceat fluidum quodvis ad id gravitans, viribus centripetis quae sint ad distantias SC, Sc, SA, ut CV, cv, AH, ordinatae curvae HVK. Comprimatur illud fluidum viribus ut FC, fc, AB ad easdem distantias SC, Sc, SA: sit B punctum in quo curva secat AB angulo 45°, ubi AB = AS. Sit

$$\mathrm{AS} = e\,, \quad \mathrm{AH} = a\,, \quad \mathrm{CV} = c\,, \quad \mathrm{AC} = x\,, \quad \mathrm{FC} = F\,.$$

Densitas Fluidi in A $= b$, in C $= D$.



Quoniam vis comprimentis incrementum est pondus incrementi fluidi superincumbentis, atque cum hujus incrementi pondus sit, ut ejus quantitas materiae & vis acceleratrix conjunctim, sive ut ejus densitas molis & vis acceleratrix i.e. ut $D \times c \times -\dot{x}$ erit vis comprimentis incrementum sive $\dot{F}$ ut $-Dc\dot{x}$ & Bo (ejus incrementum ad distantiam SA) ut $ab \times$ ob ergo

$$\dot{F} \,:\, \mathrm{Bo} \,::\, -Dc\dot{x} \,:\, ab \times \mathrm{bo}$$

ergo

$$\dot{F} = \frac{-Dc\dot{x} \times \mathrm{Bo}}{ab \times \mathrm{bo}} = \ (\text{quoniam Bo} = \text{bo})\ \frac{-Dc\dot{x}}{ab}\,.$$

Corol. I. Specifica fluidi gravitas in C est ad specificam ejus gravitatem in A ut FC ad CT subtangentem. Nam specifica fluidi gravitas est dati molis pondus, atque adeo composité ut densitas & vis acceleratrix: & proinde

$$\text{specifica gravitas in C ad eam in A} \,::\, Dc \,:\, ab \,::\, \dot{F} \,:\, \dot{x} \,::\, \mathrm{FC} \,:\, \mathrm{CT}\,.$$

Corol. II. Quoniam

$$\mathrm{CT} = \frac{-\mathrm{FC} \times \dot{x}}{\dot{F}} = \frac{-Fab}{Dc}$$

est CT negativa, adeoque BF convexa ad axem SK.

§II. Supponatur densitas compressioni proportionalis seu $F = D$ &

$$\dot{F} = \dot{D} = \frac{-Dc\dot{x}}{ab} = \;(b = e)\; = \frac{-Dc\dot{x}}{ae}$$

& *

$$\mathcal{F}\,\frac{-\dot{D}}{D} = \mathcal{F}\,\frac{c\dot{x}}{ae} \quad \text{ergo} \quad \frac{\mathrm{AHVC}}{ae} = \mathrm{L}, D\,.$$

(N. L denotat logarithmum.)

Corol. I. Si areae AHVC sumantur in progressione arithmetica densitates erunt in progressione geometrica.

Corol. II. Subtangentes curvae BF sunt reciproce ut vires centripetae nam

$$\mathrm{CT} = \frac{D\dot{x}}{-\dot{D}} = \frac{ae}{c} \quad \text{sive ut} \quad \frac{1}{c}\,.$$

Corol. III. $F = D$ curvae BF ordinatae erunt ut densitates.

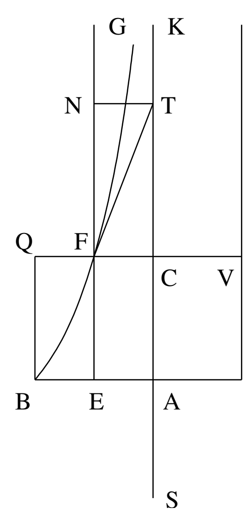
§III. Ex I. Sit vis centripeta constans seu $c = a$ &

$$\dot{D} = \frac{-Dc\dot{x}}{ae} = \frac{-D\dot{x}}{e} \quad \& \quad \frac{-\dot{D}}{D} = \frac{\dot{x}}{e}\,,$$

unde patet curvam BF esse logarithmicam asymptoto SK subtangente e descriptam.

Corol. I. Si distantiae sumantur in progressione arithmetica erunt densitates in progressione geometrica.

Corol. II. Specificae fluidi gravitates erunt in eadem progressione ($\overline{\text{nim.}}$ geometrica) nam ob datam vim centripetam, est specifica gravitas ut densitas.



* Not. $\mathcal{F}$ hic denotat fluentem quantitatem.

Corol. III. Quantitas fluidi superincumbentis in aliqua columna CFGK, quamvis infinitum spatium repleat, est ipsa tamen finita. Nam quantitas supra basin FC aequalis est quantitati homogeneae quam caperet tubus FCTN densitatis CF. Atque quantitas inter A & C super basin AB aequalis quantitati BEQF homogeniae densitatis AB.

Corol. IV. Si distantiae sumantur in progressione arithmetica erunt superincumbentes fluidi quantitates in progressione geometrica nam

$$\mathrm{GFCK} = \mathrm{FC} \times \mathrm{TN} \quad \text{sive ut} \quad \mathrm{FC}\,.$$

§IV. Ex II. Sit vis centripeta reciproce ut distantia seu $c = \dfrac{ae}{e+x}$ atque adeo

$$-\dot{D} = \frac{De\dot{x}}{e \times \overline{e+x}} = \frac{D\dot{x}}{e+x}\,,$$

ergo

$$-e\dot{D} - x\dot{D} = D\dot{x} \quad \& \quad -e\dot{D} = D\dot{x} + x\dot{D}\,,$$

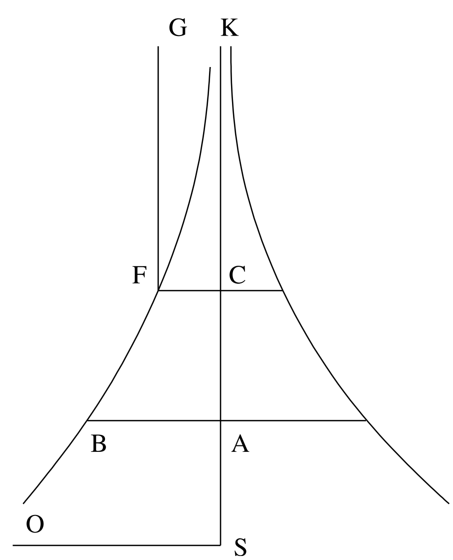
ergo fluentes quantitates inveniendo

$$e^2 - eD = Dx\,, \quad \& \quad e^2 = Dx + eD = D \times \overline{e+x}\,,$$

unde patet curvam BF esse hyperbolam aequilateram, centro S asymptotis SK, SO descriptam.

Corol. I. Si distantiae sumantur in progressione geometrica, erunt densitates, vires comprimentes & centripetae in progressione eadem suntque omnes sibi mutuo proportionales.

Corol. II. Densitates sunt reciproce ut distantiae.



Corol. III. Quantitates totales materiae in columna infinita CKGF sunt infinitae. Quod si sumantur distantiae in progressione geometrica erunt quantitates materiae in data columna in progressione arithmetica.

Corol. IV. Specificae fluidi gravitates sunt reciproce ut quadrata distantiarum nam sunt ut $Dc = \dfrac{1}{\overline{e+x}|^2}$.

Schol. Atque hac in hyp. hoce est maxime notatu dignum quod in ea sola densitas (atque adeo vis comprimens vel etiam specifica fluidi gravitas) si F = D potest esse ut quaevis potentia distantiae. Si enim D $= \dfrac{e^{m+1}}{\overline{e+x}|^m}$ erit

$$\dot{D} = \frac{-me^{m+1}\dot{x}}{\overline{e+x}|^{m+1}} = \frac{-Dc\dot{x}}{ae} = \frac{-e^{m+1}c\dot{x}}{ae \times \overline{e+x}|^m}\,,$$

$$\text{ergo} \quad \frac{m\dot{x}}{e+x} = \frac{c\dot{x}}{ae}\,, \quad \text{ergo} \quad c = \frac{mae}{e+x}\,,$$

sive reciproce ut distantia.

V. Sit universaliter

$$c = \frac{ae^n}{\overline{e+x}|^n} \quad \& \quad -\dot{D} = \frac{De^n\dot{x}}{e \times \overline{e+x}|^n}\,,$$

ergo

$$\begin{aligned} \mathcal{F}\,\frac{-\dot{D}}{D} &= \frac{-e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}} + \frac{e^{n-1}}{\overline{n-1} \times e^{n-1}} \\ &= \frac{\overline{n-1} \times \overline{e+x}|^{n-1} - \overline{n-1} \times e^{n-1}}{\overline{n-1}|^2 \times \overline{e+x}|^{n-1}} = \frac{\overline{e+x}|^{n-1} - e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}} \\ &= \mathrm{L}, D\,. \end{aligned}$$

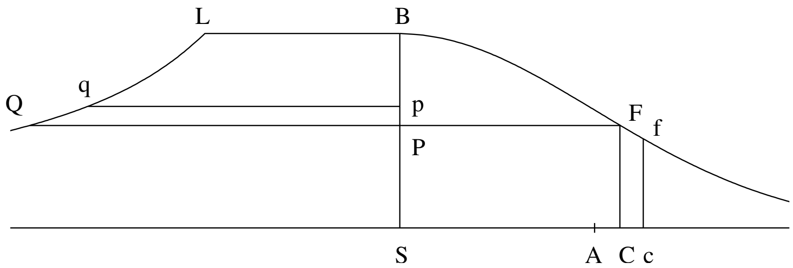
Corol. I. Ergo si $1-n$ potestas distantiae sumatur in progressione arithmetica erunt densitates in geometrica.

Corol. II. Sit vis centripeta directe ut distantia & $n = -1$ ergo $1 - n = 2$ atque adeo si quadrata distantiarum sumantur in progressione arithmetica erunt densitates in geometrica & curvae BF aequatio erit

$$\frac{-\dot{D}}{D} = \frac{\overline{e+x} \times \dot{x}}{e^2}\,,$$

quae ope hyperbolae sic construi potest. [23]

[23] MacLaurin's diagram shows the constructed curve as concave up. I believe it should be as I have drawn it. See discussion in Section 4 (equation (7)).



Descriptibatur hyperbola aequilatera LQ centro S & sit SB $= e =$ SA $=$ LB. Sumatur SC $= \sqrt{2\,\mathrm{BLQP}}$, capiatur FC $=$ SP; curva transiens per omnia puncta similiter inventa erit BF curva construenda, nam

$$\mathrm{QP} = \frac{\mathrm{SB}^2}{\mathrm{SP}} = \frac{e^2}{D} \quad \& \quad \mathrm{QPpq} = \frac{-\dot{D}e^2}{D} = \overline{e+x} \times \dot{x} = \mathrm{SC} \times \mathrm{Cc}\,,$$

ergo

$$\frac{-\dot{D}}{D} = \frac{\overline{e+x} \times \dot{x}}{e^2}\,,$$

quae erat aequatio construenda.

N. Hujus curvae subtangens est reciproce ut distantia.

Corol. III. Sit vis centripeta reciproce ut quadratum distantiae seu $c = \dfrac{ae^2}{\overline{e+x}|^2}$. Erit $n = +2$ & $1 - n = -1$ ergo si quantitates reciproce proportionales distantiis sumantur in progressione arithmetica vel ipsae distantiae in progressione musica erunt densitates in progressione geometrica; & curvae aequatio

$$\frac{-\dot{D}}{D} = \frac{e\dot{x}}{\overline{e+x}|^2}\,.$$

Quae etiam facile ope hyperbolae construi potest.

VI. Supponatur nunc vis comprimens densitatis potestati cuilibet proportionalis seu

$$F = D^m \quad \& \quad \dot{F} = mD^{m-1}\dot{D} = \frac{Dc\dot{x}}{ae}\,,$$

ergo

$$mD^{m-2}\dot{D} = \frac{c\dot{x}}{ae} \quad \& \quad \frac{mD^{m-1}}{m-1} = \mathcal{F}\,\frac{c\dot{x}}{ae}\,,$$

seu D^{m-1} ut area AHVC.

Sit porro $c = \dfrac{ae^n}{\overline{e+x}|^{n}}$ eritque $mD^{m-1}\dot{D} = \dfrac{e^{n-1}\dot{x}}{\overline{e+x}|^{n}}$, ergo

$$\frac{m}{m-1}D^{m-1} = \frac{-e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}} \quad \text{(si } b \text{ sit densitas ad distantiam } e\text{)}$$

$$+ \frac{mb^{m-1}}{m-1} + \frac{1}{n-1},$$

$$\frac{m}{m-1}D^{m-1} = \frac{mb^{m-1}}{m-1} + \frac{1}{n-1} - \frac{e^{n-1}}{\overline{n-1} \times \overline{e+x}|^{n-1}}.$$

Corol. I. Si $m = n$ erit curva cujus ordinatae sunt ut densitates hyperbola conica & densitas reciproce ut distantia.

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I— *Finis.* —T

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Minor Revision, July 2018. I am grateful to Professor David Horowitz for his comments. In particular, on p. 15 a term had been omitted in my discussion of MacLaurin's Corollary 4; this has now been rectified.